AN

INTRODUCTION TO DETERMINANTS

WITH NUMEROUS EXAMPLES

FOR THE USE OF SCHOOLS AND COLLEGES

BY

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PREFACE.

THE present work, prepared for the use of Students, and recommended for use as a text-book in the University of Edinburgh, contains the substance of the course of lectures on Determinants delivered by the author, during the Summer Sessions 1880, 1881, to the students attending the Advanced Tutorial Mathematical Class in the University.

The want of a systematic elementary treatise on Determinants has long been felt by students entering the higher departments of Mathematics. To supply, to some extent, this want, and to render an interesting and beautiful branch of Mathematical Analysis more accessible to junior students, is the object of this little treatise.

In its preparation I have freely availed myself of existing memoirs and works relating to Determinants.* The treatises to which I have been chiefly indebted are those of Dostor, Baltzer, and Mansion. I also avail myself of this opportunity of thanking Mr R. F. Scott, M.A., Fellow of St John's College, and Mr E. J. Gross, M.A., Fellow of Gonville and Caius College, Cambridge, for their kindness in permitting me to make use of the examples which have been selected from their respective works, "Theory of Determinants" and "Algebra."

While I have not hesitated to make use of what has already been written on the subject, it will be found that

* For a very full list of such works the student is referred to Scott's "Determinants," page 242, and the bibliographical notices in Baltzer's Treatise.

the present work contains much that is peculiar to it, not only in the general mode of arrangement, but in many points of detail.

One feature of interest, to which the student should pay particular attention, is the constant application of the notions of degree, and of homogeneity, which now play so important a part in modern analysis.

As my main object has been to produce a text-book suitable for beginners, many important theorems have been omitted, but the student, who masters the contents of this work, will experience no difficulty in furnishing himself with the requisite additional information from any of the more elaborate treatises.

Every important principle has been illustrated by copious examples, a considerable number of which have been fully worked out. Many of the examples are original, but the majority of them have been selected, in some cases with considerable modifications, from the works referred to above, and from the University examination papers.

Any corrections or suggestions for the improvement of the work will be thankfully received.

I have great pleasure in thanking Professors Chrystal and Tait for their kindness in examining the work in manuscript and proof-sheet, and for their many valuable suggestions for its improvement, which have in all cases been adopted. My thanks are also due to the members of the Advanced Tutorial Mathematical Class, Summer Session 1881, especially to Mr R. E. Allardice, for working through all the examples and testing the accuracy of the results.

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INTRODUCTION TO DETERMINANTS.

CHAPTER I.

DEFINITIONS, NOTATION, AND GENERAL PROPERTIES OF DETERMINANTS.

1. Preliminary Definitions.—As the notion of *degree* and *homogeneity* is of great importance, not only in the Theory of Determinants, but also in almost every branch of Mathematics, the following definitions may be useful to the student:—

Any series of letters connected merely by the signs of multiplication and division ($\times$ and $\div$) is called a *term;* for example $a \times b \times c$, $a^2 \times b \times x$, $a \times b^3 \div x$, $a^3 \times b^3 \times x^{-3}$, and $a^2 \div b^2 \div x^2$.

Any series of such terms connected by the signs of addition and subtraction ($+$ and $-$) is called a *function* of the letters involved.

Thus $a^3 + b^3 + c^3 - 3abc$ and $\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}$ are each functions of a, b, and c.

A function, which contains a finite number of terms, and involves only positive integral powers of the letters named, is called a *rational, integral, algebraic function* of these letters; for example, $3x^4 + \frac{x^3}{a^3} + \frac{x^2}{b} + 6x + c$ is a *rational,*

integral, algebraic function of x. The following are examples of functions of x which do not belong to this class:—

$$(1)\ a + \sqrt{x},\ (2)\ \frac{a}{x^3} + \frac{b}{x^2} + c,\ (3)\ e^x\ ;$$

for (1) involves a fractional power of x, (2) involves negative powers of x, since x is in the denominator of the fractions, and (3) cannot be expressed in the form of a series of multiples of positive integral powers of x, except by means of an *infinite* number of terms.

These definitions are sufficient for the present treatise; but the notion of function may be further extended.

When the expression function is used henceforth, a rational, integral, algebraic function is understood.

For the purpose of reckoning the degree of a term or of a function, letters are divided into two classes; letters taken into account in reckoning degree, called *named* or *variable* letters, and letters not taken into account in reckoning degree, called *unnamed* letters, or *constants*. In the latter class are included all numerical expressions.

The *degree of a term*, in any set of named letters, is defined to be the sum of the indices of these letters; for example, the term $6x^6y^5z^4$ is of the *fifteenth* degree in x, y, and z, of the *eleventh* degree in x and y, of the *ninth* degree in y and z, and of the *sixth* degree in x, &c.

It is here understood that a^2 stands for $a \times a$, a^3 for $a \times a \times a$, and generally a^n for $a \times a \times a \ldots\ldots\ldots n$ times.

All the ordinary laws of positive integral exponents are also assumed.

The *degree of a function*, in any set of named letters, is determined by the degree of the term of highest degree in these letters; for example, $ax + b$ is the most general function of the first degree in x. This is sometimes called a *linear* function.

Again $ax^2 + bx + c$ is a function of the second degree in x, while $ax^3 + bx^2 + cxy + dz + e$ is a function of the third degree in x, of the third degree in x and y, and of the first degree in y.

A function is said to be *homogeneous* when all its terms are of the same degree.

Thus $ax^2 + bxy + cy^2$ and $ax^2 + by^2 + cz^2 + dxy + eyz + fzx$ are each homogeneous functions of the second degree, the former in x and y, the latter in x, y, and z.

2. Preliminary Definition of Determinants.—Let us consider n^2 letters or *elements*, which we may arrange in a square of n horizontal lines or *rows*, and n vertical lines or *columns*,

thus
$$\begin{array}{cccccc} a_1 & b_1 & . & . & . & k_1 \\ a_2 & b_2 & . & . & . & k_2 \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ a_n & b_n & . & . & . & k_n \end{array};$$

and form with these a function which shall be homogeneous and of the n^{th} degree, when all the elements are considered; and homogeneous and of the first degree, when the elements of one row, or of one column, only are considered.

One term of this function, which we may call the *leading term*, will be $A\ a_1\ b_2\ c_3 \ldots\ldots\ldots k_n$ *i.e.*, the product of the elements of the dexter diagonal of the square with any co-efficient A prefixed. All other possible terms will obviously be obtained by permutating the suffixes 1, 2, 3,……n, in every possible way, and prefixing arbitrary co-efficients. The function will contain $n!$ terms, and can contain no more. It should be observed that $n!$ is always an even number, when n is not less than 2.

So far the co-efficients are arbitrary. If we impose the condition that the function be symmetrical with respect to rows and columns, *i.e.*, be unaltered, when the elements of any two rows or two columns are interchanged, a little consideration will convince the student that all the co-efficients must be equal; hence, if we add the farther condition, that the co-efficient of the leading term shall be + 1, the function will be completely determined.

The above has been taken as a simple case; in point of fact the determining conditions, in the case of the functions which form the subject of this treatise, are, that the function shall be changed in sign when two rows or two columns are interchanged, and that the co-efficient of the leading term shall be + 1. Such a function is called a *determinant.*

That the conditions of the definition do determine the function completely, may be easily shown in the particular case where $n = 2$.

Thus $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = Aa_1b_2 + Ba_2b_1$ by the first part of the definition; and

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = -\begin{vmatrix} a_2 & a_1 \\ b_2 & b_1 \end{vmatrix}$$

i.e., $$Aa_1b_2 + Ba_2b_1 = -Aa_2b_1 - Ba_1b_2$$

by the second part.

This gives

$$(A + B)(a_1b_2 + a_2b_1) = 0,$$

i.e., $$A + B = 0,$$

since a_1b_2, a_2b_1 are subject to no particular relation

Also $$A = +1.$$

Hence $$B = -1;$$

and therefore $$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

The proof might be easily generalised, but we prefer now to give the ordinary definition of a determinant, and to show that it specifies such a function as we have been describing. Considerations of the kind we have indicated show that it is the *only* rational, integral, algebraic, function, which does satisfy these conditions.

3. Formal and Ordinary Definition of Determinants.—*A determinant is a rational integral function of n^2 letters or elements, which we may arrange in a square as follows:*—

$$\begin{array}{ccccc} a_1 & b_1 & . & . & . & k_1 \\ a_2 & b_2 & . & . & . & k_2 \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ a_n & b_n & . & . & . & k_n; \end{array}$$

the first or leading term is the product of the elements in the dexter diagonal with the sign + affixed, viz., $+a_1b_2c_3\ldots\ldots\ldots k_n$, *and the others are derived from it by permuting the suffixes* 1, 2, 3,...n, *in every possible way, the signs + or − being prefixed according as the permutation is derived from the original,* 1, 2, 3,...n, *by an even or by an odd number of derangements of the suffixes.*

By a derangement is meant the placing of any smaller number after a greater, for example, $a_2b_1c_3d_4\ldots\ldots\ldots k_n$ is derived from the leading term by placing 1 after 2, *i.e.*, by one derangement; $a_2b_3c_1d_4\ldots\ldots\ldots k_n$ by first placing 1 after 2, and then 1 after 3, *i.e.*, by two derangements; the first of these terms would therefore have the sign − prefixed, the second the sign +, according to the definition.

The determinant thus defined is denoted by enclosing the square array between two vertical lines,

$$\text{thus}\qquad \begin{vmatrix} a_1 & b_1 & . & . & . & k_1 \\ a_2 & b_2 & . & . & . & k_2 \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ a_n & b_n & . & . & . & k_n \end{vmatrix};$$

or more briefly by

$$\Sigma \pm a_1 b_2 \ldots\ldots\ldots k_n.$$

4. Other Notations.—Although the single suffix notation, which we have adopted, is best suited for an elementary treatise, we may notice in passing some of the other notations in use.

Some writers employ only one symbol, and affix to it two suffixes, the first to indicate the row, and the second the column, in which the symbol occurs.

In this notation a determinant of the n^{th} order would be represented

$$\text{thus,}\ \begin{vmatrix} a_{1\cdot 1} & a_{1\cdot 2} & . & . & . & a_{1\cdot n} \\ a_{2\cdot 1} & a_{2\cdot 2} & . & . & . & a_{2\cdot n} \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ a_{n\cdot 1} & a_{n\cdot 2} & . & . & . & a_{n\cdot n} \end{vmatrix} = \Sigma \pm a_{1\cdot 1}\, a_{2\cdot 2} \ldots\ldots\ldots a_{n\cdot n}.$$

Several varieties of the double suffix notation are in use; for example—

$$\begin{vmatrix} {}^1a_1 & {}^2a_1 & . & . & . & {}^na_1 \\ {}^1a_2 & {}^2a_2 & . & . & . & {}^na_2 \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ {}^1a_n & {}^2a_n & . & . & . & {}^na_n \end{vmatrix}, \text{ and } \begin{vmatrix} {}_1a_1 & {}_2a_1 & . & . & . & {}_na_1 \\ {}_1a_2 & {}_2a_2 & . & . & . & {}_na_2 \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ {}_1a_n & {}_2a_n & . & . & . & {}_na_n \end{vmatrix}.$$

We may also omit the a's altogether, in which case the determinant would be represented thus—

$$\begin{vmatrix} (1.1) & (1.2) & \dots & (1.n) \\ (2.1) & (2.2) & \dots & (2.n) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ (n.1) & (n.2) & \dots & (n.n) \end{vmatrix}.$$

5. Expansion of Determinants by direct application of the Definition.—We will now apply our formal definition, as given in article 3, to find the value of determinants of the third and fourth orders.

Let us consider first the determinant of the third order.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Here the *leading term* is $+a_1b_2c_3$. If we now interchange the suffixes 1, 2, 3, two at a time, we obtain the five terms, $a_2b_3c_1$, $a_3b_1c_2$, $a_1b_3c_2$, $a_2b_1c_3$, and $a_3b_2c_1$. We prefix the sign + to the first two, and the sign − to the remaining three, since the former have been derived from the original permutation 1 2 3 by an even, and the latter by an odd number of derangements of the suffixes.

Hence
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{aligned} &+a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 \\ &-a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1. \end{aligned}$$

In a similar manner we have

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = \begin{aligned} &+a_1b_2c_3d_4 - a_2b_1c_3d_4 - a_1b_3c_2d_4 + a_3b_1c_2d_4 \\ &+a_2b_3c_1d_4 - a_3b_2c_1d_4 - a_1b_2c_4d_3 + a_2b_1c_4d_3 \\ &+a_1b_4c_2d_3 - a_4b_1c_2d_3 - a_2b_4c_1d_3 + a_4b_2c_1d_3 \\ &+a_1b_3c_4d_2 - a_3b_1c_4d_2 - a_1b_4c_3d_2 + a_4b_1c_3d_2 \\ &+a_3b_4c_1d_2 - a_4b_3c_1d_2 - a_2b_3c_4d_1 + a_3b_2c_4d_1 \\ &+a_2b_4c_3d_1 - a_4b_2c_3d_1 - a_3b_4c_2d_1 + a_4b_3c_2d_1. \end{aligned}$$

6. From the definition, and the examples given, it will be seen that each term of a determinant contains n elements—one and only one from each row, one and only one from each column. It is therefore homogeneous and of the n^{th} degree, when all the letters are considered, and homogeneous and of the first degree, when the elements of any one row or of any one column are considered.

7. We can also show that *the determinant is changed in sign, when two of its rows or two of its columns are interchanged.*

As a particular case, consider the determinant

$$\begin{vmatrix} a_1 & c_1 & b_1 & d_1 \\ a_2 & c_2 & b_2 & d_2 \\ a_3 & c_3 & b_3 & d_3 \\ a_4 & c_4 & b_4 & d_4 \end{vmatrix},$$

which results from interchanging the second and third columns of

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}.$$

Let Δ denote the former determinant and Δ' the latter. The leading term in Δ is $a_1c_2b_3d_4$, *i.e.*, $a_1b_3c_2d_4$, which can be derived from $a_1b_2c_3d_4$, the leading term of Δ' by interchanging the two suffixes 2 and 3, and thus the leading terms in the two determinants are equal in magnitude, but have opposite signs. In a similar manner it could be shown that every term in Δ occurs in Δ', but with an opposite sign.

Hence $\Delta = -\Delta'$.

The proof for this particular case applies equally in general.

8. It appears, therefore, from Articles 6 and 7, that the function, specified by our formal definition, possesses the properties indicated in the preliminary definition.

9. Theorem.—*A determinant remains unaltered when its rows are changed into corresponding columns, and* vice versa.

$$\text{Let } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}, \text{ and } \Delta' = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}.$$

Then $\Delta = \Delta'$.

For, in the first place, both determinants have the same elements and therefore the same terms; and in the second place, the leading term is the same in both, and hence to every term in Δ corresponds one in Δ' of the same magnitude and having the same sign.

Hence any theorem that is established concerning the rows of a determinant holds also in the case of the columns, and *vice versa.*

10. Expansion of a Determinant according to the Elements of a Single Row or Column.—From the definition we have

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$$

$$= a_1A_1 + a_2A_2 + a_3A_3,$$

where $A_1 = b_2c_3 - b_3c_2$, $A_2 = b_3c_1 - b_1c_3$, $A_3 = b_1c_2 - b_2c_1$.

Similarly

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = a_1A_1 + a_2A_2 + a_3A_3 + a_4A_4;$$

and in general, since the determinant is homogeneous, and of the first degree with respect to $a_1, a_2, a_3, \ldots a_n$, it is evident that

$$\begin{vmatrix} a_1 & b_1 & \cdot & \cdot & \cdot & k_1 \\ a_2 & b_2 & \cdot & \cdot & \cdot & k_2 \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ a_n & b_n & \cdot & \cdot & \cdot & k_n \end{vmatrix} = a_1A_1 + a_2A_2 + \ldots\ldots\ldots + a_nA_n,$$

where $A_1, A_2, \ldots\ldots A_n$ do not contain any a's, and are homogeneous functions of the $(n-1)^{th}$ degree.

We shall now proceed to prove some additional general theorems in determinants.

11. Theorem.—*If all the elements of a row or column be = 0, then the determinant is = 0.*

$$\text{For } \Delta = a_1 A_1 + a_2 A_2 + \ldots\ldots\ldots + a_n A_n = 0,$$

since $\qquad a_1, a_2, \ldots\ldots\ldots a_n$ are each $= 0$.

12. Theorem.—*If two rows or two columns be identical, the determinant vanishes.*

For, interchanging the two rows or columns that are identical, we have—

$$\Delta = -\Delta \text{ (Art. 7)},$$

or $\qquad \Delta + \Delta = 0,$

therefore $\qquad \Delta = 0.$

Corollary I.—If we interchange p pairs of rows, then

$$\Delta' = (-1)^p \Delta \text{ (Art. 7.)}$$

Similarly if we interchange q pairs of columns

$$\Delta' = (-1)^q \Delta.$$

Hence, if we interchange p pairs of rows, and q pairs of columns simultaneously, then

$$\Delta' = (-1)^{p+q} \Delta.$$

Let $p + q = m$, therefore $\Delta' = (-1)^m \Delta$. Thus Δ and Δ' have the same or opposite signs, according as m is even or odd.

For example, let $p = 1$, $q = 1$, then m is even, and hence, if we interchange any two rows and then any two columns, the resulting determinant has the same value as the original one.

Corollary II.—A *cyclical* interchange of m consecutive rows or of m consecutive columns, *i.e.*, an interchange in which each row or column is replaced by the one that follows it, and the first by the last, is equivalent to $m-1$ interchanges of pairs of rows or of columns.

Hence $\qquad \Delta' = (-1)^{m-1} \Delta$ (Cor. I.).

If we make a cyclical interchange of all the rows or columns in a determinant of the n^{th} order, then

$$\Delta' = (-1)^{n-1} \Delta.$$

Thus Δ and Δ' have the same or opposite signs according as n is odd or even.

Examples.—

(1.) $$\begin{vmatrix} a_2 & b_2 & c_2 & d_2 \\ a_1 & b_1 & c_1 & d_1 \\ a_4 & b_4 & c_4 & d_4 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix} = (-1)^2 \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}.$$

(2.) $$\begin{vmatrix} c_1 & a_1 & d_1 & b_1 \\ c_2 & a_2 & d_2 & b_2 \\ c_3 & a_3 & d_3 & b_3 \\ c_4 & a_4 & d_4 & b_4 \end{vmatrix} = (-1)^3 \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}.$$

(3.) $$\begin{vmatrix} c_2 & a_2 & d_2 & b_2 \\ c_1 & a_1 & d_1 & b_1 \\ c_4 & a_4 & d_4 & b_4 \\ c_3 & a_3 & d_3 & b_3 \end{vmatrix} = (-1)^5 \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}.$$

(4.) $$\begin{vmatrix} a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \\ a_1 & b_1 & c_1 & d_1 \end{vmatrix} = (-1)^3 \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}.$$

(5.) $$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{vmatrix} = 0.$$ (6.) $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} = 0.$$

In ex. 1 we interchange two pairs of rows, in ex. 2 three pairs of columns, in ex. 3 two pairs of rows and three pairs of columns, in ex. 4 we make a cyclical interchange of four rows, and in examples 5 and 6 there are two rows identical.

13. Theorem.—*If all the elements of a row or column be multiplied by the same factor, the determinant is multiplied by that factor.*

$$\text{Let } \Delta = \begin{vmatrix} a_1 & b_1 & \cdot & \cdot & \cdot & k_1 \\ a_2 & b_2 & \cdot & \cdot & \cdot & k_2 \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ a_n & b_n & \cdot & \cdot & \cdot & k_n \end{vmatrix} \text{ and } \Delta' = \begin{vmatrix} pa_1 & b_1 & \cdot & \cdot & \cdot & k_1 \\ pa_2 & b_2 & \cdot & \cdot & \cdot & k_2 \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ pa_n & b_n & \cdot & \cdot & \cdot & k_n \end{vmatrix},$$

then $$\Delta' = p\Delta.$$

For
$$\begin{aligned}\Delta' &= pa_1 A_1 + pa_2 A_2 + \ldots\ldots\ldots + pa_n A_n \\ &= p(a_1 A_1 + a_2 A_2 + \ldots\ldots\ldots + a_n A_n) \\ &= p\Delta.\end{aligned}$$

Corollary I.

$$\begin{vmatrix} a_1 \div p & b_1 & c_1 \\ a_2 \div p & b_2 & c_2 \\ a_3 \div p & b_3 & c_3 \end{vmatrix} = \frac{1}{p}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Corollary II.

$$\begin{vmatrix} pp^1a_1 & qp^1b & \ldots & zp^1k_1 \\ pq^1a_2 & qq^1b & \ldots & zq^1k_2 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ pz^1a_n & qz^1b_n & \ldots & zz^1k_n \end{vmatrix} = pp^1qq^1\ldots zz^1 \begin{vmatrix} a_1 & b_1 & \ldots & k_1 \\ a_2 & b_2 & \ldots & k_2 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_n & b_n & \ldots & k_n \end{vmatrix}$$

Corollary III.—*When the elements of two rows or two columns differ by a constant factor the determinant vanishes.*

Thus
$$\begin{vmatrix} pa_1 & a_1 & \ldots & k_1 \\ pa_2 & a_2 & \ldots & k_2 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ pa_n & a_n & \ldots & k_n \end{vmatrix} = p \begin{vmatrix} a_1 & a_1 & \ldots & k_1 \\ a_2 & a_2 & \ldots & k_2 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_n & a_n & \ldots & k_n \end{vmatrix} = p \times 0 = 0 \text{ (Art. 12.)}$$

Numerical example

$$\begin{vmatrix} 4 & 1 & 5 \\ 8 & 2 & 6 \\ 12 & 3 & 7 \end{vmatrix} = 4 \begin{vmatrix} 1 & 1 & 5 \\ 2 & 2 & 6 \\ 3 & 3 & 7 \end{vmatrix} = 4 \times 0 = 0.$$

Corollary IV.—*If the sign of every element in a row or a column be changed the sign of the determinant is changed.*

For this is equivalent to multiplying every element in the row or column by -1, and thus $\Delta = (-1)\ \Delta = -\Delta$ (Art. 13.)

Example

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = - \begin{vmatrix} -a_1 & b_1 & c_1 \\ -a_2 & b_2 & c_2 \\ -a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} -a_1 & -b_1 & c_1 \\ -a_2 & -b_2 & c_2 \\ -a_3 & -b_2 & c_3 \end{vmatrix} = - \begin{vmatrix} -a_1 & -b_1 & -c_1 \\ -a_2 & -b_2 & -c_2 \\ -a_3 & -b_3 & -c_3 \end{vmatrix}.$$

14.—Examples.—

(1.) $$\begin{vmatrix} 2 & 3 & 8 \\ 4 & 6 & 12 \\ 6 & 9 & 16 \end{vmatrix} = 2 \times 3 \times 4 \begin{vmatrix} 1 & 1 & 2 \\ 2 & 2 & 3 \\ 3 & 3 & 4 \end{vmatrix} = 24 \times 0 = 0.$$

(2.) $$\begin{vmatrix} pa_1 & pb_1 + qb_1 & pc_1 + qc_1 + rc_1 \\ pa_2 & pb_2 + qb_2 & pc_2 + qc_2 + rc_2 \\ pa_3 & pb_3 + qb_3 & pc_3 + qc_3 + rc_3 \end{vmatrix}$$

$$= \begin{vmatrix} pa_1 & (p+q)b_1 & (p+q+r)c_1 \\ pa_2 & (p+q)b_2 & (p+q+r)c_2 \\ pa_3 & (p+q)b_3 & (p+q+r)c_3 \end{vmatrix} = p(p+q)\,(p+q+r) \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

(3.) $$\Delta = \begin{vmatrix} bc & 1 & a \\ ca & 1 & b \\ ab & 1 & c \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}.$$

For multiplying the first row by a, the second by b, and the third by c, we obtain

$$abc\,\Delta = \begin{vmatrix} abc & a & a^2 \\ abc & b & b^2 \\ abc & c & c^2 \end{vmatrix} = abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix},$$

whence the result follows on dividing each side by abc.

(4.) $$\begin{vmatrix} 1 & b & a^2 \\ a & c & a^3 \\ a^2 & d & a^4 \end{vmatrix} = a^2 \begin{vmatrix} 1 & b & 1 \\ a & c & a \\ a^2 & d & a^2 \end{vmatrix} = a^2 \times 0 = 0.$$

$$(5.)\quad \begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}.$$

$$(6.)\quad \begin{vmatrix} bcd & a & a^2 & a^3 \\ cda & b & b^2 & b^3 \\ dab & c & c^2 & c^3 \\ abc & d & d^2 & d^3 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 & a^4 \\ 1 & b^2 & b^3 & b^4 \\ 1 & c^2 & c^3 & c^4 \\ 1 & d^2 & d^3 & d^4 \end{vmatrix}.$$

$$(7.)\quad \begin{vmatrix} a^n & b & a^{m+n} \\ a^{2n} & c & a^{m+2n} \\ a^{3n} & d & a^{m+3n} \end{vmatrix} = 0. \qquad (8.)\quad \begin{vmatrix} x^2 & xy & x^3+x^2y+x^2z \\ y^2 & yz & y^3+y^2x+y^2z \\ z^2 & zx & z^3+z^2x+z^2y \end{vmatrix} = 0.$$

15. Theorem.—*A determinant of any order whatever can always be reduced to another of the same order in which the elements of any row or any column are unity.*

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

Now multiply each element in the first column by b_1c_1, each element in the second by c_1a_1, and each element in the third by a_1b_1, and we obtain

$$a_1^2b_1^2c_1^2\Delta = \begin{vmatrix} a_1b_1c_1 & b_1c_1a_1 & c_1a_1b_1 \\ a_2b_1c_1 & b_2c_1a_1 & c_2a_1b_1 \\ a_3b_1c_1 & b_3c_1a_1 & c_3a_1b_1 \end{vmatrix} = a_1b_1c_1 \begin{vmatrix} 1 & 1 & 1 \\ a_2b_1c_1 & b_2c_1a_1 & c_2a_1b_1 \\ a_3b_1c_1 & b_3c_1a_1 & c_3a_1b_1 \end{vmatrix}.$$

Hence $a_1^2b_1^2c_1^2\Delta = a_1b_1c_1\Delta'$,

$$\text{or } \Delta = \frac{a_1b_1c_1}{a_1^2b_1^2c_1^2}\Delta' = \frac{\Delta'}{a_1b_1c_1}.$$

Determinants of any order may be treated in a similar manner.

This theorem, which is of great importance in the reduction of certain determinants, is due to Dostor.

As an example, let

$$\Delta = \begin{vmatrix} 3 & 4 & 6 \\ 2 & 8 & 8 \\ 6 & 7 & 9 \end{vmatrix}.$$

The least common multiple of 3, 4, and 6 is 12. Hence multiply the first column by 4, the second by 3, and the third by 2, and we obtain

$$4 \times 3 \times 2\ \Delta = \begin{vmatrix} 12 & 12 & 12 \\ 8 & 24 & 16 \\ 24 & 21 & 18 \end{vmatrix} = 12 \begin{vmatrix} 1 & 1 & 1 \\ 8 & 24 & 16 \\ 24 & 21 & 18 \end{vmatrix},$$

$$\text{hence } \Delta = \tfrac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ 8 & 24 & 16 \\ 24 & 21 & 18 \end{vmatrix} = 4 \times 3 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 8 & 7 & 6 \end{vmatrix}.$$

The same method was applied in example 3, Art. 14.

Corollary.—*If the first element in the leading diagonal be zero, the determinant can be reduced to one in which all the elements in the first row and first column, except the first element, are equal to unity.*

$$\text{Let } \Delta = \begin{vmatrix} 0 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

then

$$b_1c_1\ \Delta = \begin{vmatrix} 0 & b_1c_1 & b_1c_1 \\ a_2 & b_2c_1 & b_1c_2 \\ a_3 & b_3c_1 & b_1c_3 \end{vmatrix} = b_1c_1 \begin{vmatrix} 0 & 1 & 1 \\ a_2 & b_2c_1 & b_1c_2 \\ a_3 & b_3c_1 & b_1c_3 \end{vmatrix},$$

$$\text{hence } \Delta = \begin{vmatrix} 0 & 1 & 1 \\ a_2 & b_2c_1 & b_1c_2 \\ a_3 & b_3c_1 & b_1c_3 \end{vmatrix}.$$

$$\text{Again } a_2a_3\ \Delta = \begin{vmatrix} 0 & 1 & 1 \\ a_2a_3 & b_2c_1a_3 & b_1c_2a_3 \\ a_2a_3 & b_3c_1a_2 & b_1c_3a_2 \end{vmatrix} = a_2a_3 \begin{vmatrix} 0 & 1 & 1 \\ 1 & b_2c_1a_3 & b_1c_2a_3 \\ 1 & b_3c_1a_2 & b_1c_3a_2 \end{vmatrix}.$$

$$\text{and hence } \Delta = \begin{vmatrix} 0 & 1 & 1 \\ 1 & b_2c_1a_3 & b_1c_2a_3 \\ 1 & b_3c_1a_2 & b_1c_3a_2 \end{vmatrix}.$$

Examples.—

(1.)
$$\begin{vmatrix} 0 & a & b & c \\ a & 0 & c & b \\ b & c & 0 & a \\ c & b & a & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 \\ 1 & c^2 & 0 & a^2 \\ 1 & b^2 & a^2 & 0 \end{vmatrix}.$$

(2.)
$$\begin{vmatrix} 0 & a\alpha & b\beta & c\gamma \\ a\alpha & 0 & c\gamma & b\beta \\ b\beta & c\gamma & 0 & a\alpha \\ c\gamma & b\beta & a\alpha & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & c^2\gamma^2 & b^2\beta^2 \\ 1 & c^2\gamma^2 & 0 & a^2\alpha^2 \\ 1 & b^2\beta^2 & a^2\alpha^2 & 0 \end{vmatrix}.$$

16. Theorem.—*If each of the elements of one row or of one column be resolvable into m elements, then the given determinant can be exhibited as the algebraic sum of m determinants.*

$$\text{Let } \Delta = \begin{vmatrix} \alpha_1 + \alpha_2 & b_1 & c_1 \\ \beta_1 + \beta_2 & b_2 & c_2 \\ \gamma_1 + \gamma_2 & b_3 & c_3 \end{vmatrix},$$

then $\Delta = (\alpha_1 + \alpha_2)A_1 + (\beta_1 + \beta_2)A_2 + (\gamma_1 + \gamma_2)A_3$ (Art. 10)

$$= (\alpha_1A_1 + \beta_1A_2 + \gamma_1A_3) + (\alpha_2A_1 + \beta_2A_2 + \gamma_2A_3)$$

$$= \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \beta_1 & b_2 & c_2 \\ \gamma_1 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_2 & b_1 & c_1 \\ \beta_2 & b_2 & c_2 \\ \gamma_2 & b_3 & c_3 \end{vmatrix}.$$

Thus the given determinant has been resolved into the sum of two determinants, whose first columns are $\alpha_1, \beta_1, \gamma_1$, and $\alpha_2, \beta_2, \gamma_2$ respectively.

General Demonstration.

Let $\Delta = \begin{vmatrix} a_1 & b_1 & . & . & . & k_1 \\ a_2 & b_2 & . & . & . & k_2 \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ a_n & b_n & . & . & . & k_n \end{vmatrix}$, where $\begin{aligned} a_1 &= \alpha_1 + \alpha_2 + \ldots + \alpha_m, \\ a_2 &= \beta_1 + \beta_2 + \ldots + \beta_m, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ a_n &= \nu_1 + \nu_2 + \ldots + \nu_m, \end{aligned}$

then $\Delta = (\alpha_1 + \alpha_2 + \ldots + \alpha_m)A_1 + (\beta_1 + \beta_2 + \ldots\ldots + \beta_m)A_2$
$+ \ldots\ldots\ldots + (\nu_1 + \nu_2 + \ldots\ldots\ldots + \nu_m)A_n.$

Multiplying out and collecting, we obtain

$$\begin{aligned} \Delta &= (\alpha_1 A_1 + \beta_1 A_2 + \ldots\ldots\ldots + \nu_1 A_n) \\ &\quad + (\alpha_2 A_1 + \beta_2 A_2 + \ldots\ldots\ldots + \nu_2 A_n) \\ &\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\quad + (\alpha_m A_1 + \beta_m A_2 + \ldots\ldots\ldots + \nu_m A_n), \\ &= \Delta' + \Delta'' + \&c. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

Hence the given determinant has been resolved into m determinants, whose first columns are respectively

$$\begin{aligned} &\alpha_1, \beta_1, \ldots\ldots\ldots \nu_1, \\ &\alpha_2, \beta_2, \ldots\ldots\ldots \nu_2, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\alpha_m, \beta_m, \ldots\ldots\ldots \nu_m. \end{aligned}$$

Corollary I.—The proposition evidently holds when any of the elements are negative, since we have considered the expression *sum* in the algebraic sense.

Thus

$$\begin{vmatrix} \alpha_1 - \alpha_2 & a_1 & b_1 \\ \beta_1 - \beta_2 & a_2 & b_2 \\ \gamma_1 - \gamma_2 & a_2 & b_2 \end{vmatrix} = \begin{vmatrix} \alpha_1 & a_1 & b_1 \\ \beta_1 & a_2 & b_2 \\ \gamma_1 & a_3 & b_3 \end{vmatrix} + \begin{vmatrix} -\alpha_2 & a_1 & b_1 \\ -\beta_2 & a_2 & b_2 \\ -\gamma_2 & a_3 & b_3 \end{vmatrix}$$

$$= \begin{vmatrix} \alpha_1 & a_1 & b_1 \\ \beta_1 & a_2 & b_2 \\ \gamma_1 & a_3 & b_3 \end{vmatrix} - \begin{vmatrix} \alpha_2 & a_1 & b_1 \\ \beta_2 & a_2 & b_2 \\ \gamma_2 & a_3 & b_3 \end{vmatrix}.$$

Corollary II.—If the elements in the first column can be resolved into the sum of m elements, the elements in the second column into the sum of n elements, and so on, then the determinant can be resolved into $m \times n \times \ldots\ldots\ldots$ determinants.

For example—

$$\begin{vmatrix} \alpha_1 + \alpha_2 & \gamma_1 + \gamma_2 \\ \beta_1 + \beta_2 & \delta_1 + \delta_2 \end{vmatrix}$$

can be resolved into the four determinants

$$\begin{vmatrix} \alpha_1 & \gamma_1 \\ \beta_1 & \delta_1 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \gamma_2 \\ \beta_1 & \delta_2 \end{vmatrix} + \begin{vmatrix} \alpha_2 & \gamma_1 \\ \beta_2 & \delta_1 \end{vmatrix} + \begin{vmatrix} \alpha_2 & \gamma_2 \\ \beta_2 & \delta_2 \end{vmatrix}.$$

17. By means of this proposition we can sometimes simplify certain determinants by reducing them to others of equivalent value, but with smaller elements.

Thus we have

$$\begin{vmatrix} 36 & 14 & 2 & 3 \\ 21 & 18 & 2 & 1 \\ 18 & 16 & 1 & 2 \\ 12 & 8 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 22+14 & 14 & 2 & 3 \\ 3+18 & 18 & 2 & 1 \\ 2+16 & 16 & 1 & 2 \\ 4+8 & 8 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 22 & 14 & 2 & 3 \\ 3 & 18 & 2 & 1 \\ 2 & 16 & 1 & 2 \\ 4 & 8 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 14 & 14 & 2 & 3 \\ 18 & 18 & 2 & 1 \\ 16 & 16 & 1 & 2 \\ 8 & 8 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 22 & 14 & 2 & 3 \\ 3 & 18 & 2 & 1 \\ 2 & 16 & 1 & 2 \\ 4 & 8 & 0 & 1 \end{vmatrix},$$

since the second of these two determinants vanishes by Art. 12.

Similarly $\begin{vmatrix} 5 & 4 & 3 & 6 \\ 3 & 0 & 15 & 4 \\ 2 & 1 & 18 & 2 \\ 5 & 3 & 6 & 6 \end{vmatrix} = 6 \begin{vmatrix} 1 & 4 & 1 & 3 \\ 3 & 0 & 5 & 2 \\ 1 & 1 & 6 & 1 \\ 2 & 3 & 2 & 3 \end{vmatrix}.$

18. Theorem.—*The sum of m determinants, which differ only by one row or one column, can be expressed as a single determinant.*

This is merely the converse of the preceding theorem, and can be proved at once by reversing step by step the proof there given. We leave this to the student.

For example

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} 0 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} 0 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 + 0 + 0 & b_1 & c_1 \\ 0 + a_2 + 0 & b_2 & c_2 \\ 0 + 0 + a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

19. Theorem.—*If the elements of one row or column be respectively equal to the sum of the corresponding elements of other rows or columns multiplied respectively by constant factors, the determinant vanishes.*

Let $\Delta = \begin{vmatrix} \alpha_1 & b_1 & \cdot & \cdot & \cdot & k_1 \\ \alpha_2 & b_2 & \cdot & \cdot & \cdot & k_2 \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \alpha_n & b_n & \cdot & \cdot & \cdot & k_n \end{vmatrix}$, where $\begin{array}{l} \alpha_1 = pb_1 + qc_1 + \ldots + zk_1 \\ \alpha_2 = pb_2 + qc_2 + \ldots + zk_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \alpha_n = pb_n + qc_n + \ldots + zk_n, \end{array}$

then $\Delta' = \begin{vmatrix} pb_1 & b_1 & \cdot & \cdot & \cdot & k_1 \\ pb_2 & b_2 & \cdot & \cdot & \cdot & k_2 \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ pb_n & b_n & \cdot & \cdot & \cdot & k_n \end{vmatrix} + \begin{vmatrix} qc_1 & b_1 & \cdot & \cdot & \cdot & k_1 \\ qc_2 & b_2 & \cdot & \cdot & \cdot & k_2 \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ qc_n & b_n & \cdot & \cdot & \cdot & k_n \end{vmatrix}$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots + \begin{vmatrix} zk_1 & b_1 & \cdot & \cdot & \cdot & k_1 \\ zk_2 & b_2 & \cdot & \cdot & \cdot & k_2 \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ zk_n & b_n & \cdot & \cdot & \cdot & k_n \end{vmatrix}.$$

But each of these determinants vanishes by Art. 13, Cor. 3, and therefore $\Delta = 0$.

Example—

$$\begin{vmatrix} 10 & 4 & 1 \\ 14 & 3 & 4 \\ 14 & 2 & 5 \end{vmatrix} = \begin{vmatrix} 2\times 4 + 2\times 1 & 4 & 1 \\ 2\times 3 + 2\times 4 & 3 & 4 \\ 2\times 2 + 2\times 5 & 2 & 5 \end{vmatrix} = 0.$$

20. Theorem.—*A determinant remains unaltered, when we add to the elements of any row or column the corresponding elements of any of the other rows or columns multiplied respectively by constant factors.*

$$\text{Thus } \Delta = \begin{vmatrix} a_1 + pb_1 + qc_1 + \dots\dots + zk_1 & b_1 & c_1 & . & . & . & k_1 \\ a_2 + pb_2 + qc_2 + \dots\dots + zk_2 & b_2 & c_2 & . & . & . & k_2 \\ . & . & . & & & & . \\ . & . & . & & & & . \\ . & . & . & & & & . \\ a_n + pb_n + qc_n + \dots\dots + zk_n & b_n & c_n & . & . & . & k_n \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 & . & . & . & k_1 \\ a_2 & b_2 & c_2 & . & . & . & k_2 \\ . & . & . & & & & . \\ . & . & . & & & & . \\ . & . & . & & & & . \\ a_n & b_n & c_n & . & . & . & k_n \end{vmatrix} + \begin{vmatrix} pb_1 + \dots\dots + zk_1 & b_1 & c_1 & . & . & . & k_1 \\ pb_2 + \dots\dots + zk_2 & b_2 & c_2 & . & . & . & k_2 \\ . & . & . & & & & . \\ . & . & . & & & & . \\ . & . & . & & & & . \\ pb_n + \dots\dots + zk_n & b_n & c_n & . & . & . & k_n \end{vmatrix};$$

but the second of these determinants vanishes by Art. 19,

$$\text{and therefore } \Delta = \begin{vmatrix} a_1 & b_1 & . & . & . & k_1 \\ a_2 & b_2 & . & . & . & k_2 \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ a_n & b_n & . & . & . & k_n \end{vmatrix}.$$

Corollary I.—This process once performed may be repeated on the *new determinant*, thereby obtained, and so on any number of times; or the determinant obtained as the result of all these operations may be written down at once.

For example

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+b_1+c_1 & b_1+c_1 & c_1 \\ a_2+b_2+c_2 & b_2+c_2 & c_2 \\ a_3+b_3+c_3 & b_3+c_3 & c_3 \end{vmatrix}$$

may be obtained as the result of two such steps.

Similarly

$$\begin{vmatrix} ma_1+pb_1+qc_1 & nb_1+lc_1 & rc_1 \\ ma_2+pb_2+qc_2 & nb_2+lc_2 & rc_2 \\ ma_3+pb_3+qc_3 & nb_3+lc_3 & rc_3 \end{vmatrix} = m \times n \times r \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

The student must, however, beware of error in writing down the result of several steps, without going through the intermediate work.

Thus

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ is not equal to } \begin{vmatrix} a_1+b_1 & b_1+a_1 & c_1 \\ a_2+b_2 & b_2+a_2 & c_2 \\ a_3+b_3 & b_3+a_3 & c_3 \end{vmatrix}.$$

For the latter determinant vanishes since two of its columns are identical, but the former does not necessarily vanish.

In point of fact, the second determinant would not be derived from the first by any combination of single steps such as the rule indicates.

Corollary II.—Since we can add any multiple of any row or column to any other, it follows that *we can also subtract any multiple of any row or column from any other,*

without altering the value of the determinant. This transformation occurs very often in the reduction of determinants.

21. Examples—

(1.)
$$\begin{vmatrix} 4 & 5 & 6 & 7 \\ 17 & 18 & 27 & 28 \\ 21 & 22 & 31 & 33 \\ 29 & 30 & 48 & 50 \end{vmatrix} = \begin{vmatrix} 4 & 5-4 & 6 & 7-6 \\ 17 & 18-17 & 27 & 28-27 \\ 21 & 22-21 & 31 & 33-31 \\ 29 & 30-29 & 48 & 50-48 \end{vmatrix}$$

$$= \begin{vmatrix} 4 & 1 & 6 & 1 \\ 17 & 1 & 27 & 1 \\ 21 & 1 & 31 & 2 \\ 29 & 1 & 48 & 2 \end{vmatrix} = \begin{vmatrix} 4 & 1 & 6 & 1 \\ 17-4 & 1-1 & 27-6 & 1-1 \\ 21-4 & 1-1 & 31-6 & 2-1 \\ 29-4 & 1-1 & 48-6 & 2-1 \end{vmatrix}$$

$$= \begin{vmatrix} 4 & 1 & 6 & 1 \\ 13 & 0 & 21 & 0 \\ 17 & 0 & 25 & 1 \\ 25 & 0 & 42 & 1 \end{vmatrix} = - \begin{vmatrix} 1 & 4 & 6 & 1 \\ 0 & 13 & 21 & 0 \\ 0 & 17 & 25 & 1 \\ 0 & 25 & 42 & 1 \end{vmatrix} = - \begin{vmatrix} 1 & 4 & 2 & 1 \\ 0 & 13 & 5 & 0 \\ 0 & 17 & 9 & 1 \\ 0 & 25 & 8 & 1 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & 2 & 1 \\ 0 & 3 & 5 & 0 \\ 0 & -1 & 9 & 1 \\ 0 & 9 & 8 & 1 \end{vmatrix}.$$

(2.)
$$\begin{vmatrix} 9 & 13 & 17 & 4 \\ 18 & 28 & 33 & 8 \\ 30 & 40 & 54 & 13 \\ 24 & 37 & 46 & 11 \end{vmatrix} = - \begin{vmatrix} 0 & 7 & 2 & 1 \\ 0 & 2 & -1 & 0 \\ 3 & 1 & 3 & 1 \\ 3 & 2 & 5 & 1 \end{vmatrix}.$$

(3.)
$$\Delta = \begin{vmatrix} 1 & x & y+z \\ 1 & y & z+x \\ 1 & z & x+y \end{vmatrix} = 0.$$

For adding the second column to the third we get

$$\Delta = \begin{vmatrix} 1 & x & x+y+z \\ 1 & y & x+y+z \\ 1 & z & x+y+z \end{vmatrix} = (x+y+z)\begin{vmatrix} 1 & x & 1 \\ 1 & y & 1 \\ 1 & z & 1 \end{vmatrix} = (x+y+z) \times 0 = 0.$$

(4.)
$$\begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix} = (x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ z & x & y \\ y & z & x \end{vmatrix}.$$

(5.) $$\begin{vmatrix} x & x^2 & xy+xz \\ y & y^2 & yx+yz \\ z & z^2 & zx+zy \end{vmatrix} = xyz \begin{vmatrix} 1 & x & y+z \\ 1 & y & z+x \\ 1 & z & x+y \end{vmatrix} = xyz \times 0 = 0.$$

(6.) $$\begin{vmatrix} x & x+x^2 & x^2+xy+xz \\ y & y+y^2 & y^2+yz+yx \\ z & z+z^2 & z^2+zx+zy \end{vmatrix} = 0$$

(7.) $$\Delta = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (ab+bc+ca) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}.$$

For $$\Delta = \begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix},$$ (Ex. 5, Art. 14.)

and $$\begin{vmatrix} ab+ac & a & a^2 \\ bc+ba & b & b^2 \\ ca+cb & c & c^2 \end{vmatrix} = 0$$ (Ex. 5.)

therefore $$\Delta = \begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} + \begin{vmatrix} ab+ac & a & a^2 \\ bc+ba & b & b^2 \\ ca+cb & c & c^2 \end{vmatrix}$$

$$= \begin{vmatrix} ab+bc+ca & a & a^2 \\ ab+bc+ca & b & b^2 \\ ab+bc+ca & c & c^2 \end{vmatrix}$$ (Art. 18.)

$$= (ab+bc+ca) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}.$$

(8.) $$\begin{vmatrix} abc+abd+acd & a & a^2 & a^3 \\ bcd+bda+bac & b & b^2 & b^3 \\ cda+cab+cbd & c & c^2 & c^3 \\ dab+dbc+dca & d & d^2 & d^3 \end{vmatrix} = 0.$$

(9.) $$\begin{vmatrix} 1 & a^2 & a^3 & a^4 \\ 1 & b^2 & b^3 & b^4 \\ 1 & c^2 & c^3 & c^4 \\ 1 & d^2 & d^3 & d^4 \end{vmatrix} = (abc+abd+acd+bcd) \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}.$$

(See Ex. 8, and also Ex. 6, Art. 14.)

(10.) $$\begin{vmatrix} -a-b-c & a+b+c & -3a-3b-3c & -3a \\ a+b-c & a-b-c & a+\ b+\ c & a \\ a+c-b & b-c-a & a+\ b+\ c & a \\ b+c-a & c-a-b & a+\ b+\ c & a \end{vmatrix} = 0.$$

(11.) $$\begin{vmatrix} m+n-y+z & -y+z-l & -y+z-l \\ -z+x-m & n+l-z+x & -z+x-m \\ -x+y-n & -x+y-n & l+m-x+y \end{vmatrix} = 0.$$

(12). $$\begin{vmatrix} a+b+2c+ca-bc & ca-bc-a-b & ac-bc-a-b \\ ab-b-c-ac & b+c+2a+ab-ac & ab-ca-b-c \\ bc-ab-a-c & bc-ab-a-c & a+c+2b-ab+bc \end{vmatrix} = 0.$$

22. Theorem.—*If in any determinant which is a function of x, n rows or n columns become identical when $x = a$, then the determinant is divisible by $(x-a)^{n-1}$.*

As a particular case let

$$\Delta = \begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix}.$$

Subtracting the last row from the first and second we obtain

$$\Delta = \begin{vmatrix} x-a & a-a & a-x \\ a-a & x-a & a-x \\ a & a & x \end{vmatrix} = (x-a)^2 \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ a & a & x \end{vmatrix}.$$

To take a more general case let

$$\Delta = \begin{vmatrix} \phi_1(x) & \psi_1(x) & \chi_1(x) \\ \phi_2(x) & \psi_2(x) & \chi_2(x) \\ \phi_3(x) & \psi_3(x) & \chi_3(x) \end{vmatrix},$$

and let its three columns become identical when $x = a$.

By subtracting the last column from the first and second columns we have

$$\Delta = \begin{vmatrix} \phi_1(x)-\chi_1(x) & \psi_1(x)-\chi_1(x) & \chi_1(x) \\ \phi_2(x)-\chi_2(x) & \psi_2(x)-\chi_2(x) & \chi_2(x) \\ \phi_3(x)-\chi_3(x) & \psi_3(x)-\chi_3(x) & \chi_3(x) \end{vmatrix}.$$

Now when $x = a$, each of the expressions $\phi_1(a) - \chi_1(a)$, $\phi_2(a) - \chi_2(a)$, and $\phi_3(a) - \chi_3(a)$ is equal to zero, since the three original columns of Δ are identical when $x = a$, and hence $\phi_1(x) - \chi_1(x)$, $\phi_2(x) - \chi_2(x)$, and $\phi_3(x) - \chi_3(x)$ are each divisible by $x - a$. (See Kelland's Algebra, Art. 51). It can be shown in a similar manner that $\psi_1(x) - \chi_1(x)$, $\psi_2(x) - \chi_2(x)$, and $\psi_3(x) - \chi_3(x)$ are each divisible by $x - a$. Therefore the given determinant is divisible by $(x - a)^2$.

Corollary.—The theorem still holds if the rows or columns are only *proportional*, when $x = a$.

For example we have

$$\begin{vmatrix} px & qx & ra \\ px & qa & rx \\ pa & qx & rx \end{vmatrix} = p \times q \times r(x - a)^2 \begin{vmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ a & x & x \end{vmatrix}.$$

23. Examples—

(1.)
$$\Delta = \begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = (x + 2a)\ (x - a)^2.$$

For adding the second and third rows to the first we have

$$\Delta = \begin{vmatrix} x + 2a & x + 2a & x + 2a \\ a & x & a \\ a & a & x \end{vmatrix} = (x + 2a) \begin{vmatrix} 1 & 1 & 1 \\ a & x & a \\ a & a & x \end{vmatrix}.$$

Therefore $x + 2a$ is a factor of the determinant. Also we have already seen that $(x - a)^2$ is another factor. Hence $\Delta = P\ (x + 2a)\ (x - a)^2$, where P does not contain x or a, since Δ and $(x + 2a)\ (x - a)^2$ are of the same degree. Again the co-efficient of x^3 in Δ is unity, therefore $P = 1$ and $\Delta = (x + 2a)\ (x - a)^2$.

Hence generally

(2.) $$\begin{vmatrix} x & a & a & \dots & a \\ a & x & a & \dots & a \\ & & & & \\ a & a & a & \dots & x \end{vmatrix} \begin{matrix} (n \text{ rows}) \\ = \left\{x + (n-1)a\right\}\left\{x - a\right\}^{n-1}. \end{matrix}$$

(3.) $$\begin{vmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix} = (x+3)(x-1)^3.$$

(4.) $$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a).$$

For if we put $a = b$ we get two columns identical, therefore $a - b$ is a factor of Δ. Similarly for $b - c$, and $c - a$.

Thus $\Delta = P(a-b)(b-c)(c-a)$, where P cannot contain any literal factors, since $(a-b)(b-c)(c-a)$ and Δ are both of the same degree.

By equating the co-efficients of bc^2 on the two sides we have

$$P = +1,$$

and hence $$\Delta = (a-b)(b-c)(c-a).$$

(5.)

Prove that $\Delta = \begin{vmatrix} S & P & P \\ P & S & P \\ P & P & S \end{vmatrix}$, where $\begin{matrix} S = x^2 + y^2 + z^2, \\ P = xy + yz + zx, \end{matrix}$

is a complete square, and find its value.

We have as in example (1) $\Delta = (S - P)^2 (S + 2P)$

$$= (x^2 + y^2 + z^2 - xy - yz - zx)^2 (x^2 + y^2 + z^2 + 2xy + 2yz + 2zx)$$

$$= (x^2 + y^2 + z^2 - xy - yz - zx)^2 (x + y + z)^2$$

$$= \left\{(x^2 + y^2 + z^2 - xy - yz - zx)(x + y + z)\right\}^2$$

$$= (x^3 + y^3 + z^3 - 3xyz)^2.$$

(6.) In a similar manner it can be shown that

$$\begin{vmatrix} xS+(y+z)P & yS+(z+x)P & zS+(x+y)P \\ yS+(z+x)P & zS+(x+y)P & xS+(y+z)P \\ zS+(x+y)P & xS+(y+z)P & yS+(z+x)P \end{vmatrix}$$
$$= -(x^3+y^3+z^3-3xyz)^3,$$

where S and P have the same values as in the preceding example.

CHAPTER II.

MINORS AND THE REDUCTION OF DETERMINANTS.

24. Determinant Minors.—If, in any determinant, we omit any number of rows and the same number of columns, we can form a new determinant with the elements of the remaining rows and columns. Such a determinant is called a *minor* of the original determinant.

The minor formed by omitting *one* row and *one* column is called a first minor; that formed by omitting *two* rows and *two* columns a second minor; and, generally, that formed by omitting r rows and r columns an r^{th} minor.

If the original determinant be of the n^{th} order, it is evident that a first minor is a determinant of the $(n-1)^{th}$ order, a second minor a determinant of the $(n-2)^{th}$ order, and an r^{th} minor a determinant of the $(n-r)^{th}$ order.

$$\text{Let } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix},$$

$$\text{then } \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix}, \quad \begin{vmatrix} a_1 & c_1 & d_1 \\ a_3 & c_3 & d_3 \\ a_4 & c_4 & d_4 \end{vmatrix}, \text{ and } \begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix}$$

are examples of first minors each of the $(4-1)^{th}$, *i.e.*, third order.

In the first we omit the first row and first column, in the

second the second row and second column, and in the third the second row and first column.

Similarly

$$\begin{vmatrix} c_3 & d_3 \\ c_4 & d_4 \end{vmatrix}, \quad \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, \text{ and } \begin{vmatrix} a_3 & b_3 \\ a_4 & b_4 \end{vmatrix}$$

are examples of second minors, each of the second, *i.e.*, $(4 - 2)^{th}$ order.

In the first we omit the first and second rows and the first and second columns; in the second the first and fourth rows and the first and fourth columns; and in the third the first and second rows and the third and fourth columns.

25. Number of Minors.—In a determinant of the n^{th} order, there are in all n^2 first minors. For we can select one row from n rows in n different ways, and one column from n columns in n different ways, and hence we can omit one row and one column in $n \times n = n^2$ different ways.

Again, there are $\dfrac{n^2(n-1)^2}{1^2.2^2}$ second minors. For we can omit two rows from n rows in $\dfrac{n(n-1)}{1.2}$ different ways, and two columns from n columns in $\dfrac{n(n-1)}{1.2}$ different ways, and therefore two rows and two columns in $\dfrac{n(n-1)}{1.2} \times \dfrac{n(n-1)}{1.2}$ $= \dfrac{n^2(n-1)^2}{1^2.2^2}$ different ways.

In general there are $\dfrac{n^2(n-1)^2(n-2)^2 \ . \ . \ . \ (n-r+1)^2}{1^2.2^2.3^2 \ . \ . \ . \ r^2.}$ r^{th} minors in a determinant of the n^{th} order.

26. Complementary Minors.—In the determinant of the fourth order considered above, we have seen that

$$\begin{vmatrix} c_3 & d_3 \\ c_4 & d_4 \end{vmatrix}$$

is one of its second minors. If we now take the two rows and two columns omitted in forming this minor, and take the elements common to them, we can form with these elements a new determinant, which is called the *complementary minor* of the original minor;

for example $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is the complementary minor of $\begin{vmatrix} c_3 & d_3 \\ c_4 & d_4 \end{vmatrix}$.

In the determinant of the fourth order already referred to, the following are additional examples of complementary minors :—

$$|a_1| \text{ and } \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix},$$

$$|d_4| \text{ and } \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

$$\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \text{ and } \begin{vmatrix} a_3 & d_3 \\ a_4 & d_4 \end{vmatrix},$$

$$\begin{vmatrix} a_1 & d_1 \\ a_4 & d_4 \end{vmatrix} \text{ and } \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}.$$

27. Reduction of Determinants.—We have already seen that we have, from the definition, the relation

$$\triangle = a_1 A_1 + a_2 A_2 + \quad . \quad . \quad . \quad . \quad + a_n A_n.$$

We will now show that the factors A_1, A_2, . . . A_n are merely the complementary minors of, a_1, a_2, . . . a_n, with the proper signs attached.

First consider a determinant of the second order.

We have (Art. 2)

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1,$$

where b_2 b_1 are the complementary minors of a_1 a_2 respectively.

Next consider a determinant of the third order. By Art. 5 we have

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{aligned} &= a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1 \\ &= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1). \end{aligned}$$

But $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} = b_2c_3 - b_3c_2$, &c.,

therefore $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix},$

where $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, $\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$, and $\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$

are the complementary minors of a_1, a_2, and a_3 respectively.

A similar method might be applied to a determinant of the fourth order, but we prefer to give the following proof, which is evidently quite general.

We have

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = a_1A_1 + a_2A_2 + a_3A_3 + a_4A_4.$$

Now, since a_1A_1 is a function of the fourth degree, it follows that A_1 is a homogeneous function of the third degree, when all the elements in Δ, except those in the first row and first column are considered; and homogeneous

and of the first degree, when the elements of one row, or of one column, in the arrangement

$$\begin{matrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4, \end{matrix}$$

only are considered.

Hence it is evident that

$$a_1 \times \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix}$$

will furnish all the terms in $\triangle$ which have a_1 as a factor, and no more.

Hence

$$A_1 = \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix}.$$

Again

$$\begin{vmatrix} a_2 & b_2 & c_2 & d_2 \\ a_1 & b_1 & c_1 & d_1 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = -a_1A_1 - a_2A_2 - a_3A_3 - a_4A_4 \text{ (Art. 7)};$$

and by reasoning as above we obtain

$$A_2 = -\begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix}.$$

Similarly it can be shown that

$$A_3 = \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_4 & c_4 & d_4 \end{vmatrix},$$

and

$$A_4 = -\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}.$$

A moment's consideration will convince the student that the signs of the minors must be alternately + and −.

Hence we have

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix}$$

$$+ a_3 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_4 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$$

$$= a_1 \left\{ b_2 \begin{vmatrix} c_3 & d_3 \\ c_4 & d_4 \end{vmatrix} - b_3 \begin{vmatrix} c_2 & d_2 \\ c_4 & d_4 \end{vmatrix} + b_4 \begin{vmatrix} c_2 & d_2 \\ c_3 & d_3 \end{vmatrix} \right\}$$

$$- a_2 \left\{ b_1 \begin{vmatrix} c_3 & d_3 \\ c_4 & d_4 \end{vmatrix} - b_3 \begin{vmatrix} c_1 & d_1 \\ c_4 & d_4 \end{vmatrix} + b_4 \begin{vmatrix} c_1 & d_1 \\ c_3 & d_3 \end{vmatrix} \right\}$$

$$+ a_3 \left\{ b_1 \begin{vmatrix} c_2 & d_2 \\ c_4 & d_4 \end{vmatrix} - b_2 \begin{vmatrix} c_1 & d_1 \\ c_4 & d_4 \end{vmatrix} + b_4 \begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix} \right\}$$

$$- a_4 \left\{ b_1 \begin{vmatrix} c_2 & d_2 \\ c_3 & d_3 \end{vmatrix} - b_2 \begin{vmatrix} c_1 & d_1 \\ c_3 & d_3 \end{vmatrix} + b_3 \begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix} \right\}.$$

On reducing each of these determinants of the second order, we obtain a result, which agrees with that already found for the value of the same determinant, by applying the definition directly (Art. 5.)

A similar method applies to determinants of the fifth and higher orders.

We thus see that we can express any determinant of the n^{th} order, as a sum of terms, each of which is the product of one element, and of a determinant of the $(n-1)^{th}$ order; the determinants of the $(n-1)^{th}$ order may themselves be similarly treated; and this process may be continued to any extent, until we finally reduce the given determinant to its constituent terms.

28. Mnemonical Rule for reducing Determinants of the third order.—An easy way of calculating the terms of a determinant of the third order is the following, due to Sarrus. Below the given determinant repeat the first and second rows in order, or along side of it the first and second columns in order, and we obtain the following arrangements:

$$\begin{array}{ccc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{array} \quad \text{and} \quad \begin{array}{ccccc} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{array}$$

Now take, in either arrangement, the six products of all the elements, which occur three and three diagonally (as indicated by the dotted lines in the arrangements), and affix + to those descending from left to right, and − to those ascending from left to right, and we obtain as before

$$\triangle = +a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1.$$

In practice we need only *imagine* the rows or columns repeated.

For example we have

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} \begin{array}{l} = 1.3.5 + 2.4.3 + 3.2.4 - 3.3.3 - 2.2.5 - 1.4.4 \\ = 15 + 24 + 24 - 27 - 20 - 16 = 0. \end{array}$$

29. Examples.—We now proceed to apply these results to the reduction of a number of determinants.

(1.) $\begin{vmatrix} 1 & 2 \\ 2 & 6 \end{vmatrix} = 1.6 - 2.2 = 6 - 4 = 2.$

(2.) $\begin{vmatrix} 1 & 3 \\ 4 & 0 \end{vmatrix} = 1.0 - 3.4 = -12.$

(3.) $\begin{vmatrix} a+b\sqrt{-1} & -c+d\sqrt{-1} \\ c+d\sqrt{-1} & a-b\sqrt{-1} \end{vmatrix}$

$= (a+b\sqrt{-1})(a-b\sqrt{-1}) - (c+d\sqrt{-1})(-c+d\sqrt{-1})$
$= a^2+b^2+c^2+d^2.$

(4.) $\begin{vmatrix} 4 & 1 & 5 \\ 6 & 2 & 6 \\ 8 & 3 & 4 \end{vmatrix} = 2\begin{vmatrix} 2 & 1 & 5 \\ 3 & 2 & 6 \\ 4 & 3 & 4 \end{vmatrix}$

$$= 2\left\{2\begin{vmatrix} 2 & 6 \\ 3 & 4 \end{vmatrix} - 3\begin{vmatrix} 1 & 5 \\ 3 & 4 \end{vmatrix} + 4\begin{vmatrix} 1 & 5 \\ 2 & 6 \end{vmatrix}\right\}$$
$$= 4\,(8-18) - 6\,(4-15) + 8\,(6-10)$$
$$= -40 + 66 - 32 = -6.$$

(5.) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$

$$= (b-a)\,(c-a)\begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix} = (a-b)\,(b-c)\,(c-a).$$

(6.) $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = x_1(y_2-y_3) + x_2(y_3-y_1) + x_3(y_1-y_2).$

(7.) $\begin{vmatrix} a & \gamma & \beta \\ \gamma & e & \alpha \\ \beta & \alpha & i \end{vmatrix} = aei + 2\alpha\beta\gamma - a\alpha^2 - e\beta^2 - i\gamma^2.$

The student of co-ordinate geometry should take special note of examples (6) and (7), as the former represents double the area of the triangle, the co-ordinates of whose vertices are (x_1y_1) (x_2y_2) (x_3y_3), and the latter, when equated to zero, expresses the condition that the general equation of the second degree

$$ax^2 + 2\gamma xy + ey^2 + 2\beta x + 2\alpha y + c = 0$$

may represent two right lines.

(8.) $\begin{vmatrix} 1 & 1 & 1 \\ z & x & y \\ y & z & x \end{vmatrix} = x^2 + y^2 + z^2 - xy - yz - zx.$

(9.) $\begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix} = (x + y + z) \begin{vmatrix} 1 & 1 & 1 \\ z & x & y \\ y & z & x \end{vmatrix}$ (Ex. 4, Art. 21.)

$= (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$ Ex. 8.

$= x^3 + y^3 + z^3 - 3xyz.$

(See also Exs. 5 and 6, Art. 23.)

(10.) $\begin{vmatrix} c & a & a \\ b & c & a \\ b & b & c \end{vmatrix} = a^2b + ab^2 + c^3 - 3abc.$

(11.) $\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 4.$

(12.) $\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} = 2abc.$

(13.) $\begin{vmatrix} 0 & ab^2 & ac^2 \\ ba^2 & 0 & bc^2 \\ ca^2 & cb^2 & 0 \end{vmatrix} = 2a^3b^3c^3.$

(14.) $\begin{vmatrix} 2 & 5 & 6 & 2 \\ 3 & 4 & 2 & 6 \\ 4 & 8 & 1 & 2 \\ 5 & 3 & 1 & 6 \end{vmatrix} = \begin{vmatrix} 2+5+6+2 & 5 & 6 & 2 \\ 3+4+2+6 & 4 & 2 & 6 \\ 4+8+1+2 & 8 & 1 & 2 \\ 5+3+1+6 & 3 & 1 & 6 \end{vmatrix}$

$= 15 \begin{vmatrix} 1 & 5 & 6 & 2 \\ 1 & 4 & 2 & 6 \\ 1 & 8 & 1 & 2 \\ 1 & 3 & 1 & 6 \end{vmatrix}$

$= 15 \begin{vmatrix} 1-1 & 5-3 & 6-1 & 2-6 \\ 1-1 & 4-3 & 2-1 & 6-6 \\ 1-1 & 8-3 & 1-1 & 2-6 \\ 1 & 3 & 1 & 6 \end{vmatrix} = 15 \begin{vmatrix} 0 & 2 & 5 & -4 \\ 0 & 1 & 1 & 0 \\ 0 & 5 & 0 & -4 \\ 1 & 3 & 1 & 6 \end{vmatrix}$

$$= -15\begin{vmatrix} 2 & 5 & -4 \\ 1 & 1 & 0 \\ 5 & 0 & -4 \end{vmatrix} = 60\begin{vmatrix} 2 & 5 & 1 \\ 1 & 1 & 0 \\ 5 & 0 & 1 \end{vmatrix}$$

$$= 60\left\{\begin{vmatrix} 1 & 1 \\ 5 & 0 \end{vmatrix} + \begin{vmatrix} 2 & 5 \\ 1 & 1 \end{vmatrix}\right\}$$

$$= 60\{(0-5)+(2-5)\} = -480.$$

(15.) $\begin{vmatrix} 3 & 1 & 2 & 3 \\ 4 & 0 & 2 & 1 \\ 6 & 4 & 1 & 2 \\ 7 & 3 & 0 & 1 \end{vmatrix} = 50.$ (16.) $\begin{vmatrix} 12 & 16 & 24 & 33 \\ 20 & 25 & 35 & 45 \\ 20 & 27 & 36 & 55 \\ 28 & 38 & 51 & 78 \end{vmatrix} = -20.$

(17.) $\begin{vmatrix} 25 & -15 & 23 & -5 \\ -15 & -10 & 19 & 5 \\ 23 & 19 & -15 & 9 \\ -5 & 5 & 9 & -5 \end{vmatrix} = 194400.$

(18.) $\begin{vmatrix} 5 & -10 & 11 & 0 \\ -10 & -11 & 12 & 4 \\ 11 & 12 & -11 & 2 \\ 0 & 4 & 2 & -6 \end{vmatrix} = 8100.$

(19.) $\begin{vmatrix} 0 & a & b & c \\ a & 0 & \gamma & \beta \\ b & \gamma & 0 & \alpha \\ c & \beta & \alpha & 0 \end{vmatrix} = (-a\alpha - \beta b + c\gamma)^2 - 4ab\alpha\beta.$

(20.) $\begin{vmatrix} 0 & a & a & a \\ b & 0 & a & a \\ b & b & 0 & a \\ b & b & b & 0 \end{vmatrix} = -ab\,(a^2 + ab + b^2).$

(21.) $\begin{vmatrix} -a & a & a & a \\ a & -a & a & a \\ a & a & -a & a \\ a & a & a & -a \end{vmatrix} = -16a^4.$

(22.) $$\begin{vmatrix} 0 & a & a & a \\ a & 0 & a & a \\ a & a & 0 & a \\ a & a & a & 0 \end{vmatrix} = -3a^4.$$

(23.) $$\begin{vmatrix} 0 & a & a & a \\ b & 0 & b & b \\ c & c & 0 & c \\ d & d & d & 0 \end{vmatrix} = -3abcd.$$

(24.) $$\begin{vmatrix} -a & b & c & d \\ b & -a & d & c \\ c & d & -a & b \\ d & c & b & -a \end{vmatrix} = \begin{array}{c} a^4 + b^4 + c^4 + d^4 \\ -2a^2b^2 - 2a^2c^2 - 2a^2d^2 \\ -2b^2c^2 - 2b^2d^2 - 2c^2d^2 \\ -8abcd \end{array}$$

Additional examples will be found in Chap. IV.

30. Theorem.—*The sum of the products obtained by multiplying the elements of any row or column by the corresponding minors, with their proper signs, of the elements of any other row or column, is equal to zero.*

We have already seen that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1A_1 + a_2A_2 + a_3A_3,$$

where A_1, A_2, A_3 are the complementary minors of a_1, a_2, a_3, with their proper signs attached.

But if we substitute b_1 for a_1, b_2 for a_2, and b_3 for a_3, we obtain

$$\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} = b_1A_1 + b_2A_2 + b_3A_3.$$

Now this determinant vanishes, since it has two columns identical, and hence

$$b_1A_1 + b_2A_2 + b_3A_3 = 0$$

Similarly $$c_1A_1 + c_2A_2 + c_3A_3 = 0, \text{ \&c.}$$

31. Theorem.—*If in a determinant of the n^{th} order all the elements, but one, vanish in any row or column the determinant reduces to one of the $(n - 1)^{th}$ order, multiplied by the element which does not vanish.*

This and the following theorem follow at once from Art. 10
10 ~~7~~. We have

$$\triangle = a_1A_1 + a_2A_2 + \;\ldots\; + a_nA_n\,;$$

but $$a_2, a_3, \;\ldots\; a_n \text{ all vanish,}$$

and therefore $$\triangle = a_1A_1,$$

where A_1 is a determinant with one row and one column less than $\triangle$.

For example

$$\begin{vmatrix} 1 & a & b \\ 0 & a_1 & b_1 \\ 0 & a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

Conversely, any determinant can be expressed as a determinant of a higher order without altering its value, by a suitable transformation.

It is at once evident from the above that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 1 & \alpha & \beta & \gamma \\ 0 & a_1 & b_1 & c_1 \\ 0 & a_2 & b_2 & c_2 \\ 0 & a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 1 & \lambda & \mu & \nu & \rho \\ 0 & 1 & \alpha & \beta & \gamma \\ 0 & 0 & a_1 & b_1 & c_1 \\ 0 & 0 & a_2 & b_2 & c_2 \\ 0 & 0 & a_3 & b_3 & c_3 \end{vmatrix},$$

where α, β, γ, λ, μ, ν, and ρ are any quantities whatever. This process can be carried on without limit.

32. Theorem.—*If all the elements on one side of the leading diagonal vanish, the determinant reduces to the leading term.*

Thus

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & 0 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_1 & c_2 \\ 0 & c_3 \end{vmatrix} = a_1 b_2 c_3.$$

Similarly

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ 0 & b_2 & c_2 & d_2 \\ 0 & 0 & c_3 & d_3 \\ 0 & 0 & 0 & d_4 \end{vmatrix} = a_1 b_2 c_3 d_4.$$

33. Expansion of a Determinant as a sum of products of Complementary Minors.—We have seen that we can expand

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

in terms of the elements of a single column. We will now show that we can expand the same determinant in terms of the determinant minors contained in any two columns.

In the expansion of Δ already given (Art. 27), the following two terms occur, $a_1 b_2 \begin{vmatrix} c_3 & d_3 \\ c_4 & d_4 \end{vmatrix}$, and $-a_2 b_1 \begin{vmatrix} c_3 & d_3 \\ c_4 & d_4 \end{vmatrix}$.

Combining these we get

$$(a_2b_1 - a_1b_2)\begin{vmatrix} c_3 & d_3 \\ c_4 & d_4 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} c_3 & d_3 \\ c_4 & d_4 \end{vmatrix}.$$

By combining in pairs in a similar way all the other terms of the expansion, we obtain

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} c_3 & d_3 \\ c_4 & d_4 \end{vmatrix} - \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} \times \begin{vmatrix} c_2 & d_2 \\ c_4 & d_4 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 \\ a_4 & b_4 \end{vmatrix} \times \begin{vmatrix} c_2 & d_2 \\ c_3 & d_3 \end{vmatrix}$$
$$+ \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \times \begin{vmatrix} c_1 & d_1 \\ c_4 & d_4 \end{vmatrix} - \begin{vmatrix} a_2 & b_2 \\ a_4 & b_4 \end{vmatrix} \times \begin{vmatrix} c_1 & d_1 \\ c_3 & d_3 \end{vmatrix} + \begin{vmatrix} a_3 & b_3 \\ a_4 & b_4 \end{vmatrix} \times \begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix}.$$

This result may be written thus

$$\Delta = (12)(34) - (13)(24) + (14)(23)$$
$$+ (23)(14) - (24)(13) + (34)(12),$$

where the numbers within the first pair of each set of brackets are abbreviations for the determinants whose elements are a's and b's, while those within the second pair are corresponding abbreviations for the determinants whose elements are c's and d's.

Thus $(12)(34) = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} c_3 & d_3 \\ c_4 & d_4 \end{vmatrix}$, where,

(12) precedes (34), while

$(34)(12) = \begin{vmatrix} a_3 & b_3 \\ a_4 & b_4 \end{vmatrix} \times \begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix}$, where

(12) follows (34).

34. Again we have

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 \\ a_4 & b_4 & c_4 & d_4 & e_4 \\ a_5 & b_5 & c_5 & d_5 & e_5 \end{vmatrix} = (1)(2345) - (2)(1345) + (3)(1245) - (4)(1235) + (5)(1234),$$

$$\text{where } (1)(2345) = a_1 \begin{vmatrix} b_2 & c_2 & d_2 & e_2 \\ b_3 & c_3 & d_3 & e_3 \\ b_4 & c_4 & d_4 & e_4 \\ b_5 & c_5 & d_5 & e_5 \end{vmatrix} \text{ \&c.}$$

Now expanding in a similar manner each of the determinants (2345), (1345), &c., we obtain a sum of terms, each of which contains as a factor a determinant of the third order. If we combine these terms in pairs we get

$$\begin{aligned} \Delta = {} & (123)(45) + (142)(35) + (134)(25) + (243)(15) \\ & + (125)(34) + (315)(24) + (235)(14) + (145)(23) \\ & + (425)(13) + (345)(12), \end{aligned}$$

$$\text{where } (123)(45) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} d_4 & e_4 \\ d_5 & e_5 \end{vmatrix} \text{ \&c.}$$

Similarly

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 & f_3 \\ a_4 & b_4 & c_4 & d_4 & e_4 & f_4 \\ a_5 & b_5 & c_5 & d_5 & e_5 & f_5 \\ a_6 & b_6 & c_6 & d_6 & e_6 & f_6 \end{vmatrix} \begin{aligned} = {} & (123)(456) + (124)(536) + (125)(346) \\ & + (126)(354) + (134)(256) + (135)(264) \\ & + (136)(245) + (145)(236) + (146)(253) \\ & + (156)(234) + (234)(165) + (235)(146) \\ & + (236)(154) + (245)(163) + (246)(135) \\ & + (256)(143) + (345)(126) + (346)(152) \\ & + (356)(124) + (456)(132), \end{aligned}$$

$$\text{where } (123)(456) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} d_4 & e_4 & f_4 \\ d_5 & e_5 & f_5 \\ d_6 & e_6 & f_6 \end{vmatrix} \text{ \&c.}$$

The preceding examples are particular cases of a more general theorem due to Laplace, for the statement and proof of which the student is referred to the larger treatises on Determinants.

35. Examples.—

(1.)
$$\begin{vmatrix} a_1 & a_2 & -1 & 0 \\ b_1 & b_2 & 0 & -1 \\ 0 & 0 & \alpha_1 & \beta_1 \\ 0 & 0 & \alpha_2 & \beta_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ 0 & 0 & \alpha_1 & \beta_1 \\ 0 & 0 & \alpha_2 & \beta_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}.$$

(2.)
$$\begin{vmatrix} a_1 & b_1 & 0 & 0 & 0 \\ a_2 & b_2 & 0 & 0 & 0 \\ a_3 & b_3 & 0 & 0 & 0 \\ a_4 & b_4 & c_4 & d_4 & e_4 \\ a_5 & b_5 & c_5 & d_5 & e_5 \end{vmatrix} = 0.$$

(3.)
$$\begin{vmatrix} a_1 & a_2 & a_3 & -1 & 0 & 0 \\ b_1 & b_2 & b_3 & 0 & -1 & 0 \\ c_1 & c_2 & c_3 & 0 & 0 & -1 \\ 0 & 0 & 0 & \alpha_1 & \beta_1 & \gamma_1 \\ 0 & 0 & 0 & \alpha_2 & \beta_2 & \gamma_2 \\ 0 & 0 & 0 & \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ 0 & 0 & 0 & \alpha_1 & \beta_1 & \gamma_1 \\ 0 & 0 & 0 & \alpha_2 & \beta_2 & \gamma_2 \\ 0 & 0 & 0 & \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}.$$

CHAPTER III.

MULTIPLICATION OF DETERMINANTS.

36. Product of two Determinants of the Second Order.—Let us consider in the first place the product of the two determinants

$$P = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \text{ and } Q = \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}.$$

We will now prove that

$$P \times Q = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} = \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 \\ a_2\alpha_1 + b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 \end{vmatrix} = \Delta.$$

$$\text{For } \Delta = \begin{vmatrix} a_1\alpha_1 & a_1\alpha_2 \\ a_2\alpha_1 & a_2\alpha_2 \end{vmatrix} + \begin{vmatrix} b_1\beta_1 & b_1\beta_2 \\ b_2\beta_1 & b_2\beta_2 \end{vmatrix} + \begin{vmatrix} a_1\alpha_1 & b_1\beta_2 \\ a_2\alpha_1 & b_2\beta_2 \end{vmatrix} + \begin{vmatrix} b_1\beta_1 & a_1\alpha_2 \\ b_2\beta_1 & a_2\alpha_2 \end{vmatrix}.$$

But the first two of these four determinants vanish, and

$$\text{therefore } \Delta = \begin{vmatrix} a_1\alpha_1 & b_1\beta_2 \\ a_2\alpha_1 & b_2\beta_2 \end{vmatrix} + \begin{vmatrix} b_1\beta_1 & a_1\alpha_2 \\ b_2\beta_1 & a_2\alpha_2 \end{vmatrix}$$

$$= \alpha_1\beta_2 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} + \alpha_2\beta_1 \begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix}$$

$$= (\alpha_1\beta_2 - \alpha_2\beta_1) \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$$= \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} \times \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

37. Other Forms of the Product.—In the preceding article we have shown that

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} = \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 \\ a_2\alpha_1 + b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 \end{vmatrix}. \qquad (1.)$$

Similarly
$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} = \begin{vmatrix} a_1\alpha_1 + b_1\alpha_2 & a_1\beta_1 + b_1\beta_2 \\ a_2\alpha_1 + b_2\alpha_2 & a_2\beta_1 + b_2\beta_2 \end{vmatrix} \quad (2.)$$

$$= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} = \begin{vmatrix} a_1\alpha_1 + a_2\beta_1 & a_1\alpha_2 + a_2\beta_2 \\ b_1\alpha_1 + b_2\beta_1 & b_1\alpha_2 + b_2\beta_2 \end{vmatrix} \quad (3.)$$

$$= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} = \begin{vmatrix} a_1\alpha_1 + a_2\alpha_2 & a_1\beta_1 + a_2\beta_2 \\ b_1\alpha_1 + b_2\alpha_2 & b_1\beta_1 + b_2\beta_2 \end{vmatrix} \quad (4.)$$

Hence

The product of two determinants P and Q of the second order can be expressed as a determinant of the second order in four ways in general different.

We can in fact combine as above either :—

1st. The elements of each row of Q with the corresponding elements of all the rows of P;

2nd. The elements of each row of Q with the corresponding elements of all the columns of P;

3rd. The elements of each column of Q with the corresponding elements of all the rows of P;

4th. The elements of each column of Q with the corresponding elements of all the columns of P.

38. Second Demonstration.—The proposition in Art. 36 may also be proved in the following manner :—

We have seen (example 1, Art. 35) that

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & -1 & 0 \\ b_1 & b_2 & 0 & -1 \\ 0 & 0 & \alpha_1 & \beta_1 \\ 0 & 0 & \alpha_2 & \beta_2 \end{vmatrix} = \Delta.$$

To the first column of Δ add a_1 times the third + b_1

times the fourth, then to the second column add a_2-times the third + b_2 times the fourth, and we obtain

$$\Delta = \begin{vmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ a_1\alpha_1 + b_1\beta_1 & a_2\alpha_1 + b_2\beta_1 & \alpha_1 & \beta_1 \\ a_1\alpha_2 + b_1\beta_2 & a_2\alpha_2 + b_2\beta_2 & \alpha_2 & \beta_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_2\alpha_1 + b_2\beta_1 \\ a_1\alpha_2 + b_1\beta_2 & a_2\alpha_2 + b_2\beta_2 \end{vmatrix} \times \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} \text{ (Art. 34.)}$$

$$\text{But } \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = 1$$

hence

$$\Delta = \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_2\alpha_1 + b_2\beta_1 \\ a_1\alpha_2 + b_1\beta_2 & a_2\alpha_2 + b_2\beta_2 \end{vmatrix} = \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 \\ a_2\alpha_1 + b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 \end{vmatrix},$$

which is the first form of the product given in Art. 37.

The other three forms of the product may be obtained in a similar manner. We leave the detail as an exercise to the student.

39. Product of Two Determinants of the Third Order.

Let $P = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, and $Q = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$, then

$$P \times Q = \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix} = \Delta.$$

For if we expand Δ we obtain 27 determinants, all of which will be found on trial to vanish, except the following six :—

$$(1.) \begin{vmatrix} a_1\alpha_1 & b_1\beta_2 & c_1\gamma_3 \\ a_2\alpha_1 & b_2\beta_2 & c_2\gamma_3 \\ a_3\alpha_1 & b_3\beta_2 & c_3\gamma_3 \end{vmatrix} = \alpha_1\beta_2\gamma_3 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

$$(2.)\quad \begin{vmatrix} a_1\alpha_1 & c_1\gamma_2 & b_1\beta_3 \\ a_2\alpha_1 & c_2\gamma_2 & b_2\beta_3 \\ a_3\alpha_1 & c_3\gamma_2 & b_3\beta_3 \end{vmatrix} = \alpha_1\gamma_2\beta_3 \begin{vmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{vmatrix} = -\alpha_1\beta_3\gamma_2 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

$$(3.)\quad \begin{vmatrix} c_1\gamma_1 & a_1\alpha_2 & b_1\beta_3 \\ c_2\gamma_1 & a_2\alpha_2 & b_2\beta_3 \\ c_3\gamma_1 & a_3\alpha_2 & b_3\beta_3 \end{vmatrix} = \gamma_1\alpha_2\beta_3 \begin{vmatrix} c_1 & a_1 & b_1 \\ c_2 & a_2 & b_2 \\ c_3 & a_3 & b_3 \end{vmatrix} = \alpha_2\beta_3\gamma_1 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

$$(4.)\quad \begin{vmatrix} b_1\beta_1 & a_1\alpha_2 & c_1\gamma_3 \\ b_2\beta_1 & a_2\alpha_2 & c_2\gamma_3 \\ b_3\beta_1 & a_3\alpha_2 & c_3\gamma_3 \end{vmatrix} = \beta_1\alpha_2\gamma_3 \begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} = -\alpha_2\beta_1\gamma_3 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

$$(5.)\quad \begin{vmatrix} b_1\beta_1 & c_1\gamma_2 & a_1\alpha_3 \\ b_2\beta_1 & c_2\gamma_2 & a_2\alpha_3 \\ b_3\beta_1 & c_3\gamma_2 & a_3\alpha_3 \end{vmatrix} = \beta_1\gamma_2\alpha_3 \begin{vmatrix} b_1 & c_1 & a_1 \\ b_2 & c_2 & a_2 \\ b_3 & c_3 & a_3 \end{vmatrix} = \alpha_3\beta_1\gamma_2 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

$$(6.)\quad \begin{vmatrix} c_1\gamma_1 & b_1\beta_2 & a_1\alpha_3 \\ c_2\gamma_1 & b_2\beta_2 & a_2\alpha_3 \\ c_3\gamma_1 & b_3\beta_2 & a_3\alpha_3 \end{vmatrix} = \gamma_1\beta_2\alpha_3 \begin{vmatrix} c_1 & b_1 & a_1 \\ c_2 & b_2 & a_2 \\ c_3 & b_3 & a_3 \end{vmatrix} = -\alpha_3\beta_2\gamma_1 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Hence we have

$$\Delta = \left\{\alpha_1(\beta_2\gamma_3 - \beta_3\gamma_2) + \alpha_2(\beta_3\gamma_1 - \beta_1\gamma_3) + \alpha_3(\beta_1\gamma_2 - \beta_2\gamma_1)\right\} \times \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} \times \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

The student should satisfy himself that all the other 21 determinants vanish.

For example

$$\begin{vmatrix} a_1\alpha_1 & a_1\alpha_2 & a_1\alpha_3 \\ a_2\alpha_1 & a_2\alpha_2 & a_2\alpha_3 \\ a_3\alpha_1 & a_3\alpha_2 & a_3\alpha_3 \end{vmatrix} = \alpha_1\alpha_2\alpha_3 \begin{vmatrix} a_1 & a_1 & a_1 \\ a_2 & a_2 & a_2 \\ a_3 & a_3 & a_3 \end{vmatrix} = \alpha_1\alpha_2\alpha_3 \times 0 = 0.$$

Similarly for the others.

Here we have combined the rows of P with the rows of Q. We can however combine, as in the case of determinants of the second order, the rows of P with the columns of Q, the columns of P with the rows of Q, or the columns of P with the columns of Q; and hence *the product of two determinants of the third order can be expressed as a determinant of the third order in four ways in general different.*

40. Second Demonstration..—We have (by example 3, Art. 35)

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 & -1 & 0 & 0 \\ b_1 & b_2 & b_3 & 0 & -1 & 0 \\ c_1 & c_2 & c_3 & 0 & 0 & -1 \\ 0 & 0 & 0 & \alpha_1 & \beta_1 & \gamma_1 \\ 0 & 0 & 0 & \alpha_2 & \beta_2 & \gamma_2 \\ 0 & 0 & 0 & \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = \Delta.$$

If to the first column of Δ we add the three last columns, multiplied by a_1, b_1, c_1 respectively, to the second column the three last multiplied by a_2, b_2, c_2 respectively, and to the third the last three multiplied by a_3, b_3, c_3 respectively, we obtain

$$\Delta = \begin{vmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & \alpha_1 & \beta_1 & \gamma_1 \\ a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & \alpha_2 & \beta_2 & \gamma_2 \\ a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 & \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

$$= - \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 \\ a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 \\ a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}$$

$$\times \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} \text{ (Art. 34.).}$$

Now

$$\begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = -1,$$

therefore $$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 \\ a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 \\ a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}.$$

41. Product of two Determinants of different orders.—If we have to multiply together two determinants of different orders, we first transform the one of less degree into a determinant of the same order as the other, and then apply the rule already given.

Thus

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} 1 & 0 & 0 \\ 0 & \alpha_1 & \beta_1 \\ 0 & \alpha_2 & \beta_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1\alpha_1 + c_1\beta_1 & b_1\alpha_2 + c_1\beta_2 \\ a_2 & b_2\alpha_1 + c_2\beta_1 & b_2\alpha_2 + c_2\beta_2 \\ a_3 & b_3\alpha_1 + c_3\beta_1 & b_3\alpha_2 + c_3\beta_2 \end{vmatrix}.$$

The proofs which we have given for the multiplication of determinants of the second and third orders evidently apply in general, hence we have the general proposition :—

The product of two determinants of the n^{th} and m^{th} orders respectively (n not greater than m) is a determinant of the m^{th} order, which can be expressed in four ways in general different.

42. We will now show how, by bordering the determinants in a way which does not alter their value, we can represent their product as a determinant of any order from n to $2n$ inclusive. We will prove this for the case when $n = 2$.

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} = \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 \\ a_2\alpha_1 + b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 \end{vmatrix},$$

a determinant of the second order.

Also

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ 0 & 0 & 1 \end{vmatrix} \times - \begin{vmatrix} \alpha_1 & 0 & \beta_1 \\ \alpha_2 & 0 & \beta_2 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= - \begin{vmatrix} a_1\alpha_1 & a_1\alpha_2 & b_1 \\ a_2\alpha_1 & a_2\alpha_2 & b_2 \\ \beta_1 & \beta_2 & 0 \end{vmatrix},$$

a determinant of the third order.

Again $$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & 0 & 0 \\ a_2 & b_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \times \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha_1 & \beta_1 \\ 0 & 0 & \alpha_2 & \beta_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & 0 & 0 \\ a_2 & b_2 & 0 & 0 \\ 0 & 0 & \alpha_1 & \beta_1 \\ 0 & 0 & \alpha_2 & \beta_2 \end{vmatrix}.$$

a determinant of the fourth order.

In a similar manner we can express the product of two determinants of the third order as a determinant of any order from the third to the sixth inclusive, and similarly for determinants of the n^{th} order.

43. Since $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 = \begin{vmatrix} a_1^2 + b_1^2 & a_1a_2 + b_1b_2 \\ a_1a_2 + b_1b_2 & a_2^2 + b_2^2 \end{vmatrix}$, we obtain

$$(a_1b_2 - a_2b_1)^2 = (a_1^2 + b_1^2)(a_2^2 + b_2^2) - (a_1a_2 + b_1b_2)^2,$$

that is

$$(a_1^2 + b_1^2)(a_2^2 + b_2^2) = (a_1b_2 - a_2b_1)^2 + (a_1a_2 + b_1b_2)^2;$$

the well known proposition in the theory of numbers :—

The product of two numbers each of which is the sum of two squares can always be expressed as the sum of two squares.

44. Reciprocal Determinants.—*If from any determinant Δ we form another Δ', whose elements are the minors corresponding to each element of Δ, then Δ' is called the reciprocal of Δ.*

Thus $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$ is the reciprocal of $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$,

when $A_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, $A_2 = \begin{vmatrix} b_3 & c_3 \\ b_1 & c_1 \end{vmatrix}$, &c.

If we form the product of Δ and Δ' we obtain

$$\Delta \times \Delta' = \begin{vmatrix} a_1A_1 + b_1B_1 + c_1C_1 & a_2A_1 + b_2B_1 + c_2C_1 & a_3A_1 + b_3B_1 + c_3C_1 \\ a_1A_2 + b_1B_2 + c_1C_2 & a_2A_2 + b_2B_2 + c_2C_2 & a_3A_2 + b_3B_2 + c_3C_2 \\ a_1A_3 + b_1B_3 + c_1C_3 & a_2A_3 + b_2B_3 + c_2C_3 & a_3A_3 + b_3B_3 + c_3C_3 \end{vmatrix}.$$

But $a_1A_1 + b_1B_1 + c_1C_1$, $a_2A_2 + b_2B_2 + c_2C_2$, and $a_3A_3 + b_3B_3 + c_3C_3$ are each equal to Δ (Art. 10), while $a_1A_2 + b_1B_2 + c_1C_2$, $a_2A_1 + b_2B_1 + c_2C_1$, &c., all vanish (Art. 30), and

$$\text{therefore } \Delta \times \Delta' = \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix} = \Delta^3,$$

hence $\qquad \Delta' = \Delta^2.$

And in general $\Delta \times \Delta' = \Delta^n$; *i.e.*, $\Delta' = \Delta^{n-1}$.

Hence, *the determinant whose elements are the minors corresponding to each element of a given determinant is the* $(n-1)^{th}$ *power of that determinant.*

45. Theorem.—*If* Δ *denote any determinant, then any minor of order* m, *of the reciprocal of* Δ, *is equal to the product of* Δ^{m-1} *and the complementary minor of the corresponding minor of* Δ.

First let
$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

We have successively

$$\Delta \begin{vmatrix} B_2 & C_2 \\ B_3 & C_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} 1 & 0 & 0 \\ 0 & B_2 & C_2 \\ 0 & B_3 & C_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1B_2 + c_1C_2 & b_2B_2 + c_2C_2 & b_3B_2 + c_3C_2 \\ b_1B_3 + c_1C_3 & b_2B_3 + c_2C_3 & b_3B_3 + c_3C_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ -a_1A_2 & \Delta - a_2A_2 & -a_3A_2 \\ -a_1A_3 & -a_2A_3 & \Delta - a_3A_3 \end{vmatrix}.$$

since $\qquad a_1A_1 + b_1B_1 + c_1C_1 = \Delta$ (Art. 10)

and $\qquad a_1A_2 + b_1B_2 + c_1C_2 = 0$, &c., (Art. 30).

Now the last determinant may be written thus:

$$\begin{vmatrix} a_1 + 0 & a_2 + 0 & a_3 + 0 \\ 0 - a_1A_2 & \Delta - a_2A_2 & 0 - a_3A_3 \\ 0 - a_1A_3 & 0 - a_2A_3 & \Delta - a_3A_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix} + \text{seven other determinants, each}$$

of which will be found on trial to vanish;

and therefore
$$\Delta\begin{vmatrix} B_2 & C_2 \\ B_3 & C_3 \end{vmatrix} = a_1\Delta^2,$$

or
$$\begin{vmatrix} B_2 & C_2 \\ B_3 & C_3 \end{vmatrix} = a_1\Delta^{2-1} = a_1 \times \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Similarly
$$\begin{vmatrix} B_3 & C_3 \\ B_1 & C_1 \end{vmatrix} = a_2 \times \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Again let
$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}.$$

We have as before

$$\Delta\begin{vmatrix} C_3 & D_3 \\ C_4 & D_4 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \times \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & C_3 & D_3 \\ 0 & 0 & C_4 & D_4 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1C_3 + d_1D_3 & c_2C_3 + d_2D_3 & c_3C_3 + d_3D_3 & c_4C_3 + d_4D_3 \\ c_1C_4 + d_1D_4 & c_2C_4 + d_2D_4 & c_3C_4 + d_3D_4 & c_4C_4 + d_4D_4 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ -a_1A_3 - b_1B_3 & -a_2A_3 - b_2B_3 & \Delta - a_3A_3 - b_3B_3 & -a_4A_3 - b_4B_3 \\ -a_1A_4 - b_1B_4 & -a_2A_4 - b_2B_4 & -a_3A_4 - b_3B_4 & \Delta - a_4A_4 - b_4B_4 \end{vmatrix}.$$

If we now add to the elements of the third row A_3 times the elements of the first row $+ B_3$ times the elements of the second row, and to the fourth row A_4 times the first $+ B_4$ times the second, we obtain

$$\Delta \begin{vmatrix} C_3 & D_3 \\ C_4 & D_4 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ 0 & 0 & \Delta & 0 \\ 0 & 0 & 0 & \Delta \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \times \begin{vmatrix} \Delta & 0 \\ 0 & \Delta \end{vmatrix} \text{ (Art. 34)}$$

$$= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \Delta^2,$$

and hence

$$\begin{vmatrix} C_3 & D_3 \\ C_4 & D_4 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \Delta^{2-1} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \times \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}.$$

Similarly it may be shown that

$$\begin{vmatrix} B_2 & C_2 & D_2 \\ B_3 & C_3 & D_3 \\ B_4 & C_4 & D_4 \end{vmatrix} = a_1 \times \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}^2.$$

We have proved the theorem for two particular cases. The method is, however, perfectly general.

Corollary.—*If a determinant vanish, its minors A_1, A_2, &c., are respectively proportional to B_1, B_2, &c.*

For example we have as above

$$\begin{vmatrix} B_2 & C_2 \\ B_3 & C_3 \end{vmatrix} = a_1 \Delta ;$$

but if $\Delta = 0$, then $\begin{vmatrix} B_2 & C_2 \\ B_3 & C_3 \end{vmatrix} = a_1 \times 0 = 0,$

and hence $B_2C_3 - B_3C_2 = 0$

or $\dfrac{B_2}{B_3} = \dfrac{C_2}{C_3}.$

In general if

$$\begin{vmatrix} a_1 & b_1 & . & . & . & k_1 \\ a_2 & b_2 & . & . & . & k_2 \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ a_n & b_n & . & . & . & k_n \end{vmatrix} = 0$$

then

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} = \&c.,$$

$$\frac{A_2}{A_3} = \frac{B_2}{B_3} = \frac{C_2}{C_3} = \&c.,$$

$$\&c.,$$

46. Multiplication of Matrices.—The notation

$$\begin{Vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{Vmatrix}$$

(where the number of columns is greater than the number of rows) is used to denote the three determinants

$$\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \quad \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}, \text{ and } \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix},$$

obtained by omitting in turn each of the columns. The system $\begin{Vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{Vmatrix}$ is sometimes called a *Matrix.*

Let us now consider two such systems

$$\begin{Vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{Vmatrix}, \text{ and } \begin{Vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{Vmatrix}.$$

If we take the sum of the products of every possible determinant which can be formed out of the one system by the corresponding determinant of the other system, we obtain as result

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \times \begin{vmatrix} \beta_1 & \gamma_1 \\ \beta_2 & \gamma_2 \end{vmatrix} + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} \times \begin{vmatrix} \gamma_1 & \alpha_1 \\ \gamma_2 & \alpha_2 \end{vmatrix}.$$

But the determinant $\begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 \end{vmatrix}$

(formed by combining each row of the system

$$\left\| \begin{matrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{matrix} \right\| \text{ with all the rows of } \left\| \begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{matrix} \right\|),$$

$$= \begin{vmatrix} a_1\alpha_1 & a_1\alpha_2 \\ a_2\alpha_1 & a_2\alpha_2 \end{vmatrix} + \begin{vmatrix} a_1\alpha_1 & b_1\beta_2 \\ a_2\alpha_1 & b_2\beta_2 \end{vmatrix} + \&c.$$

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \times \begin{vmatrix} \beta_1 & \gamma_1 \\ \beta_2 & \gamma_2 \end{vmatrix} + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} \times \begin{vmatrix} \gamma_1 & \alpha_1 \\ \gamma_2 & \alpha_2 \end{vmatrix}.$$

Thus the law for multiplication of ordinary determinants applies also to matrices.

Corollary.—As a particular example, let $\alpha_1 = a_1$, $\beta_1 = b_1$, $\gamma_1 = c_1$, &c., and we obtain

$$\begin{vmatrix} a_1^2 + b_1^2 + c_1^2 & a_1a_2 + b_1b_2 + c_1c_2 \\ a_1a_2 + b_1b_2 + c_1c_2 & a_2^2 + b_2^2 + c_2^2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2,$$

i.e. $(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2) = (a_1b_2 - a_2b_1)^2 + (a_1c_2 - a_2c_1)^2$
$+ (b_1c_2 - b_2c_1)^2 + (a_1a_2 + b_1b_2 + c_1c_2)^2.$

Hence the product of two numbers, each of which is the sum of three squares, can be expressed as the sum of four squares.

Next let $a_1 = b_1 = c_1 = 1$, and $a_2 = a$, $b_2 = b$, $c_2 = c$, and we obtain

$$\begin{vmatrix} 1 + 1 + 1 & a + b + c \\ a + b + c & a^2 + b^2 + c^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ a & b \end{vmatrix}^2 + \begin{vmatrix} 1 & 1 \\ b & c \end{vmatrix}^2 + \begin{vmatrix} 1 & 1 \\ c & a \end{vmatrix}^2,$$

or $\begin{vmatrix} a^0 + b^0 + c^0 & a + b + c \\ a + b + c & a^2 + b^2 + c^2 \end{vmatrix} = (b - a)^2 + (c - b)^2 + (a - c)^2,$

a result which is of importance in the Theory of Algebraic Equations.

Again, if we compound in a similar manner the two matrices

$$\left\| \begin{matrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{matrix} \right\| \text{ and } \left\| \begin{matrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{matrix} \right\|,$$

we obtain

$$\begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 & a_1\alpha_3 + b_1\beta_3 \\ a_2\alpha_1 + b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 & a_2\alpha_3 + b_2\beta_3 \\ a_3\alpha_1 + b_3\beta_1 & a_3\alpha_2 + b_3\beta_2 & a_3\alpha_3 + b_3\beta_3 \end{vmatrix} = 0,$$

since, if we expand this determinant, each of its component determinants vanishes identically.

47. We conclude this chapter by proving Euler's Theorem.—*The product of two numbers, each of which is the sum of four squares, can be expressed as the sum of four squares.*

We have by multiplication

$$\begin{vmatrix} a & b \\ -b_1 & a_1 \end{vmatrix} \times \begin{vmatrix} c & d \\ -d_1 & c_1 \end{vmatrix} = \begin{vmatrix} ac + bd & -ad_1 + bc_1 \\ -b_1c + a_1d & b_1d_1 + a_1c_1 \end{vmatrix}.$$

Now let

$$\begin{aligned} a &= x + y\sqrt{-1}, & a_1 &= x - y\sqrt{-1}, \\ b &= u + v\sqrt{-1}, & b_1 &= u - v\sqrt{-1}, \\ c &= p + q\sqrt{-1}, & c_1 &= p - q\sqrt{-1}, \\ d &= r + s\sqrt{-1}, & d_1 &= r - s\sqrt{-1}. \end{aligned}$$

Substitute these values and reduce the three determinants and there results

$$\begin{aligned} (x^2 + y^2 + u^2 + v^2)(p^2 + q^2 + r^2 + s^2) &= (px - qy + ru - sv)^2 \\ &+ (py + qx + rv + su)^2 + (pu + qv - rx - sy)^2 \\ &+ (pv - qu - ry + sx)^2. \end{aligned}$$

This result may be expressed in a great variety of ways (See Scott's Determinants, Art. 18, Chap. VI.).

CHAPTER IV.

MISCELLANEOUS EXAMPLES AND DETERMINANTS OF SPECIAL FORMS.

48. Miscellaneous Examples—

(1) $$\Delta = \begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix} = (a-b)(a-c)(a-d)(b-c)(b-d)(c-d).$$

For the determinant vanishes if we put $a = b$ and therefore $a - b$ is a factor of it. Similarly for the other factors $a - c$, $a - d$, &c.

Thus $\Delta = P(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$, where P cannot contain any factor involving a, b, c, or d, since the expression which multiplies P is of the same degree as Δ. Now the co-efficient of bc^2d^3 is $+1$ on the left and P on the right, hence $P = +1$,

and therefore $\Delta = (a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$.

We can perform the reduction by another method. For subtract the last column in turn from the other three columns and we obtain successively,

$$\Delta = \begin{vmatrix} 0 & 0 & 0 & 1 \\ a-d & b-d & c-d & d \\ a^2-d^2 & b^2-d^2 & c^2-d^2 & d^2 \\ a^3-d^3 & b^3-d^3 & c^3-d^3 & d^3 \end{vmatrix} = -\begin{vmatrix} a-d & b-d & c-d \\ a^2-d^2 & b^2-d^2 & c^2-d^2 \\ a^3-d^3 & b^3-d^3 & c^3-d^3 \end{vmatrix}$$

$$= -(a-d)(b-d)(c-d)\begin{vmatrix} 1 & 1 & 1 \\ a+d & b+d & c+d \\ a^2+ad+d^2 & b^2+bd+d^2 & c^2+cd+d^2 \end{vmatrix}$$

$$= -(a-d)(b-d)(c-d)\begin{vmatrix} 0 & 0 & 1 \\ a-c & b-c & c+d \\ a^2-c^2+ad-cd & b^2-c^2+bd-cd & c^2+cd+d^2 \end{vmatrix}$$

$$= -(a-d)(b-d)(c-d)\begin{vmatrix} a-c & b-c \\ (a-c)(a+c+d) & (b-c)(b+c+d) \end{vmatrix}$$

$$= -(a-d)(b-d)(c-d)(a-c)(b-c)\begin{vmatrix} 1 & 1 \\ a+c+d & b+c+d \end{vmatrix}$$

$$= -(a-d)(b-d)(c-d)(a-c)(b-c)\{b+c+d-a-c-d\}$$

$$= (a-b)(a-c)(a-d)(b-c)(b-d)(c-d).$$

The following nine examples may be treated in a similar manner :—

(2.) $$\begin{vmatrix} 1 & 1 & 1 \\ ab & bc & ca \\ a^2b^2 & b^2c^2 & c^2a^2 \end{vmatrix} = abc(a-b)(b-c)(a-c).$$

(3.) $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a+b+c)(a-b)(b-c)(c-a).$$

(4.) $$\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (ab+bc+ca)(a-b)(b-c)(c-a).$$

(5.) $$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^4 & b^4 & c^4 & d^4 \end{vmatrix} = (a+b+c+d)(a-b)(a-c)(a-d)(b-c)(b-d)(c-d).$$

(6.) $$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^3 & b^3 & c^3 & d^3 \\ a^4 & b^4 & c^4 & d^4 \end{vmatrix} = (ab+ac+ad+bc+bd+cd)(a-b)(a-c)(a-d)(b-c)(b-d)(c-d).$$

(7.) $$\begin{vmatrix} b^2c^2d^2 & 1 & a & a(a-1) \\ c^2d^2a^2 & 1 & b & b(b-1) \\ d^2a^2b^2 & 1 & c & c(c-1) \\ a^2b^2c^2 & 1 & d & d(d-1) \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \\ a^4 & b^4 & c^4 & d^4 \end{vmatrix}$$

$= \big(abc + abd + acd + bcd\big)(a-b)(a-c)\big(a-d\big)\big(b-c\big)$ $\big(b-d\big)\big(c-d\big)$. (See ex. 1 and also ex. 9, Art. 21.)

(8.) $$\begin{vmatrix} 1 & 1 & 1 & \cdots \\ a & b & c & \cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ a^{n-1} & b^{n-1} & c^{n-1} & \cdots \end{vmatrix}$$

$= (a-b)(a-c)\ldots\ldots \times (b-c)(b-d)\ldots\ldots \times (c-d)$, &c.

(9.) $$\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ \frac{1}{a^2} & \frac{1}{b^2} & \frac{1}{c^2} \end{vmatrix} = \frac{(a-b)\,(b-c)\,(a-c)}{a^2b^2c^2}.$$

(10.) $$\begin{vmatrix} \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ \frac{1}{a^2} & \frac{1}{b^2} & \frac{1}{c^2} \\ \frac{1}{a^3} & \frac{1}{b^3} & \frac{1}{c^3} \end{vmatrix} = \frac{(a-b)\,(b-c)\,(a-c)}{a^3b^3c^3}.$$

The preceding determinants are merely particular cases of the form

$$\begin{vmatrix} \phi(x) & \psi(x) & \chi(x) & \cdots \\ \phi(y) & \psi(y) & \chi(y) & \cdots \\ \phi(z) & \psi(z) & \chi(z) & \cdots \\ \cdot\,\cdot & \cdot\,\cdot & \cdot\,\cdot & \cdots \end{vmatrix},$$

where $\phi(x)$, $\psi(x)$, $\chi(x)$ denote *rational, integral, algebraic*

functions of x, and $\varphi(y)$, &c., $\varphi(z)$, &c., the same functions of y, z respectively. For a discussion of this general determinant the student is referred to Salmon's Higher Algebra, note to ex. 5, Art. 21.

(11.) $$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1+a_1 & 1 & \cdots & 1 \\ 1 & 1 & 1+a_2 & \cdots & 1 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 1 & 1 & 1 & \cdots & 1+a_n \end{vmatrix} = a_1 a_2 a_3 \ldots\ldots a_n.$$

For if any one of the quantities $a_1, a_2, \ldots\ldots a_n$ vanishes, then the determinant vanishes, since it then has two columns identical, and therefore it is divisible by $a_1\, a_2\, a_3 \ldots\ldots a_n$.

Thus $$\triangle = P a_1 a_2 a_3 \ldots\ldots a_n,$$

when P is obviously $+1$.

(12.) $$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix} = xy.$$

(13.) $$\begin{vmatrix} 1+a_1 & 1 & 1 & \cdots & 1 \\ 1 & 1+a_2 & 1 & \cdots & 1 \\ 1 & 1 & 1+a_3 & \cdots & 1 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 1 & 1 & 1 & \cdots & 1+a_n \end{vmatrix}$$

$$= a_1 a_2 \ldots\ldots a_n \left\{ 1 + \frac{1}{a_1} + \frac{1}{a_2} + \ldots\ldots + \frac{1}{a_n} \right\}.$$

For if any one of the quantities $a_1, a_2, \ldots\ldots a_n$ vanishes, then the determinant is reduced to a case of ex. (11). Thus by putting $a_1 = 0$, $a_2 = 0$ we obtain successively the

terms $a_2a_3a_4\ldots\ldots a_n$, $a_1a_3a_4\ldots\ldots a_n$, &c. If we now consider the product of the dexter diagonal elements, we see that $a_1a_2a_3\ldots\ldots a_n$ is also a term in the result.

The following three examples are particular cases of the preceding :—

(14.) $$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

(15.) $$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix} = a^3\left(1+\frac{3}{a}\right) = a^2(a+3).$$

(16.) $$\begin{vmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 1+\overline{1-3} & 1 & 1 & 1 \\ 1 & 1+\overline{1-3} & 1 & 1 \\ 1 & 1 & 1+\overline{1-3} & 1 \\ 1 & 1 & 1 & 1+\overline{1-3} \end{vmatrix}$$

$$= (1-3)^3\,(1-3+4) = -16.$$

Examples (11) to (16) inclusive may also be treated by successive reduction to determinants of lower order. We leave this to the student.

(17.) $$\Delta = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = (a+b+c)\,(a+b\omega^2+c\omega)\,(a+b\omega+c\omega^2),$$
where $\omega^3 = 1$.

For we have already seen that, by adding the second and third rows to the first, $a+b+c$ is a factor.

Again it is at once evident, since $\omega^3 = 1$, $\omega^6 = 1$, &c., that

$$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = \begin{vmatrix} a & b\omega^2 & c\omega \\ c\omega & a & b\omega^2 \\ b\omega^2 & c\omega & a \end{vmatrix} = (a+b\omega^2+c\omega)\begin{vmatrix} 1 & 1 & 1 \\ c\omega & a & b\omega^2 \\ b\omega^2 & c\omega & a \end{vmatrix}.$$

Thus $a + b\omega^2 + c\omega$ is a second factor of $\triangle$.

Also $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = \begin{vmatrix} a & b\omega & c\omega^2 \\ c\omega^2 & a & b\omega \\ b\omega & c\omega^2 & a \end{vmatrix} = (a + b\omega + c\omega^2) \begin{vmatrix} 1 & 1 & 1 \\ c\omega^2 & a & b\omega \\ b\omega & c\omega^2 & a \end{vmatrix}$,

and hence $a + b\omega + c\omega^2$ is another factor of $\triangle$.

Thus $\triangle = P(a + b + c)(a + b\omega^2 + c\omega)(a + b\omega + c\omega^2)$, where P is obviously $+1$, as we see by equating the co-efficients of a^3.

The above may be proved more simply thus:—

$$\triangle = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = (a+b+c)(a^2+b^2+c^2-ab-bc-ca).$$ (Ex. 9, Art 29.)

Thus $a + b + c$ is a factor of $\triangle$, *i.e.*, of $a^3 + b^3 + c^3 - 3abc$. Hence also $a + b\omega^2 + c\omega$ is a factor of $a^3 + b^3\omega^6 + c^3\omega^3 - 3abc\omega^3$, *i.e.*, of $a^3 + b^3 + c^3 - 3abc$, since $\omega^3 = 1$, and $\omega^6 = 1$.

Similarly it can be shown that $a + b\omega + c\omega^2$ is a factor.

We may also adopt the following more general method. We have by Art. 20

$$\triangle = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = \begin{vmatrix} a+\lambda c+\mu b & b+\lambda a+\mu c & c+\lambda b+\mu c \\ c & a & b \\ b & c & a \end{vmatrix}.$$

Hence $a + \lambda c + \mu b$ is a factor of $\triangle$, if $b + \lambda a + \mu c$ and $c + \lambda b + \mu c$ be each divisible by it, *i.e.*, if

$$\mu = \frac{1}{\lambda} = \frac{\lambda}{\mu}, \text{ or } \lambda^3 = 1 \text{ and } \mu = \lambda^2.$$

Thus $a + \lambda^2 b + \lambda c$ is a factor of $\triangle$. Similarly it may be shown that $a + \lambda b + \lambda^2 c$ is a factor.

Hence as before

$$\triangle = (a + b + c)(a + \lambda^2 b + \lambda c)(a + \lambda b + \lambda^2 c).$$

It may also be shown that

$$\begin{aligned}\Delta &= (a+b+c)(\lambda a+b+\lambda^2 c)(\lambda^2 a+b+\lambda c)\\ &= (a+b+c)(\lambda a+\lambda^2 b+c)(\lambda^2 a+\lambda b+c).\end{aligned}$$

We leave this as an exercise to the student.

We will now give one or two additional illustrations of the method of resolving determinants into linear factors.

(18.)
$$\Delta = \begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix} = -(a+b+c+d)(a+b-c-d)(a+d-b-c)(d+b-a-c).$$

For we have successively

$$\Delta = \begin{vmatrix} a+b+c+d & a+b+c+d & a+b+c+d & a+b+c+d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix};$$

$$= (a+b+c+d)\begin{vmatrix} 1 & 1 & 1 & 1 \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix}$$

$$= (a+b+c+d)\begin{vmatrix} 1-1 & 1-1 & 1-1 & 1 \\ b-c & a-c & d-c & c \\ c-b & d-b & a-b & b \\ d-a & c-a & b-a & a \end{vmatrix}$$

$$= -(a+b+c+d)\begin{vmatrix} b-c & a-c & d-c \\ c-b & d-b & a-b \\ d-a & c-a & b-a \end{vmatrix}.$$

Now add the second column to the first and subtract the third and we obtain

$$\Delta = -(a+b+c+d)\begin{vmatrix} a+b-c-d & a-c & d-c \\ -(a+b-c-d) & d-b & a-b \\ -(a+b-c-d) & c-a & b-a \end{vmatrix}$$

$$= -(a+b+c+d)(a+b-c-d)\begin{vmatrix} 1 & a-c & d-c \\ -1 & d-b & a-b \\ -1 & c-a & b-a \end{vmatrix}$$

$$= -(a+b+c+d)(a+b-c-d)\begin{vmatrix} 1 & a-c & d-c \\ 0 & a+d-b-c & a+d-b-c \\ 0 & 0 & d+b-a-c \end{vmatrix}$$

$$= -(a+b+c+d)(a+b-c-d)(a+d-b-c)(b+d-c-a)\begin{vmatrix} 1 & a-c & d-c \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= -(a+b+c+d)(a+b-c-d)(a+d-b-c)(b+d-c-a).$$

This example may also be treated in the following manner :—

To the first column add the other three and we get $a+b+c+d$ as a factor of the determinant. Again to the first column add the other three multiplied respectively by 1, -1 and -1, and we have $a+b-c-d$ as another factor. Similarly we can show that $a+d-b-c$ and $a+c-d-b$ are factors of the determinant.

Hence, since the co-efficient of a^4 is unity, we have at once $\triangle = (a+b+c+d)(a+b-c-d)(a+d-b-c)(a+c-d-b)$.

Corollary.—If $a=0$ we have

$$(19.)\quad \begin{vmatrix} 0 & b & c & d \\ b & 0 & d & c \\ c & d & 0 & b \\ d & c & b & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & d^2 & c^2 \\ 1 & d^2 & 0 & b^2 \\ 1 & c^2 & b^2 & 0 \end{vmatrix}$$

$$= -(b+c+d)(c+d-b)(b+d-c)(b+c-d)$$
$$= (d^2+b^2-c^2)^2 - 4d^2b^2$$
$$= d^4+b^4+c^4-2d^2b^2-2b^2c^2-2c^2d^2.$$

$$(20.)\begin{vmatrix} 0 & a & b & c \\ a & 0 & c_1 & b_1 \\ b & c_1 & 0 & a_1 \\ c & b_1 & a_1 & 0 \end{vmatrix} = -(\sqrt{aa_1}+\sqrt{bb_1}+\sqrt{cc_1})\\(\sqrt{aa_1}+\sqrt{bb_1}-\sqrt{cc_1})\\(-\sqrt{aa_1}+\sqrt{bb_1}+\sqrt{cc_1})\\(\sqrt{aa_1}-\sqrt{bb_1}+\sqrt{cc_1})$$

$$= (aa_1+bb_1-cc_1)^2 - 4aa_1bb_1$$
$$= a^2a_1{}^2 + b^2b_1{}^2 + c^2c_1{}^2 - 2aa_1bb_1 - 2bb_1cc_1 - 2cc_1aa_1.$$

$$(21.)\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & a & b \\ 1 & a & 0 & c \\ 1 & b & c & 0 \end{vmatrix} = a^2+b^2+c^2-2ab-2bc-2ca \\ = (a+b-c)^2 - 4ab \\ = -(\sqrt{a}+\sqrt{b}+\sqrt{c})(\sqrt{a}+\sqrt{b}-\sqrt{c}) \\ (\sqrt{a}-\sqrt{b}+\sqrt{c})(-\sqrt{a}+\sqrt{b}+\sqrt{c}).$$

$$(22.)\quad \Delta = \begin{vmatrix} -a & b & c & d \\ b & -a & d & c \\ c & d & -a & b \\ d & c & b & -a \end{vmatrix} = -(b+c+d-a)(c+d+a-b)\\(d+a+b-c)(a+b+c-d).$$

$$(23.)\quad \Delta = \begin{vmatrix} b^2+c^2 & ab & ca \\ ab & c^2+a^2 & bc \\ ca & bc & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2.$$

For multiplying the first row by a, the second by b, and the third by c, we obtain

$$\Delta = \frac{1}{abc}\begin{vmatrix} ab^2+ac^2 & a^2b & a^2c \\ ab^2 & a^2b+bc^2 & b^2c \\ ac^2 & bc^2 & a^2c+b^2c \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} 0 & -2bc^2 & -2b^2c \\ ab^2 & a^2b+bc^2 & b^2c \\ ac^2 & bc^2 & a^2c+b^2c \end{vmatrix}$$

$$= \frac{4a^3b^3c^3}{abc} = 4a^2b^2c^2.$$

$$(24.)\begin{vmatrix} (a+b)^2 & c^2 & c^2 \\ a^2 & (b+c)^2 & a^2 \\ b^2 & b^2 & (c+a)^2 \end{vmatrix} = 2abc(a+b+c)^3.$$

(25.) $$\begin{vmatrix} (a+b)^2 & bc & ac \\ bc & (a+c)^2 & ab \\ ac & ab & (b+c)^2 \end{vmatrix} = 2abc(a+b+c)^3.$$

(26.) $$\begin{vmatrix} \dfrac{b+c}{a} & \dfrac{a}{b+c} & \dfrac{a}{b+c} \\ \dfrac{b}{a+c} & \dfrac{a+c}{b} & \dfrac{b}{a+c} \\ \dfrac{c}{a+b} & \dfrac{c}{a+b} & \dfrac{a+b}{c} \end{vmatrix} = \frac{2(a+b+c)^3}{(a+b)(b+c)(c+a)}.$$

49. Conjugate Elements.—Elements are said to be *conjugate* to each other, when, considered with reference to their respective rows and columns, they hold the same positions.

Thus in

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

a_2 is conjugate to b_1, a_3 to c_1, and b_3 to c_2. The conjugates of a_1, b_2, and c_3 are evidently a_1, b_2, and c_3 respectively.

It is easily seen that conjugate elements occupy the position of a point and its image considered with respect to a plane mirror coinciding with the principal diagonal.

50. Symmetrical, Skew, and Skew Symmetrical Determinants.—When the conjugate elements of a determinant are equal to each other, the determinant is said to be *symmetrical*; for example

$$\begin{vmatrix} a & b & d \\ b & c & e \\ d & e & f \end{vmatrix} \text{ and } \begin{vmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{vmatrix}.$$

Determinants, in which each element, *except those in the principal diagonal,* is equal to its conjugate with its sign changed, are called *skew* determinants; for example

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ -b_1 & b_2 & c_2 & d_2 \\ -c_1 & -c_2 & c_3 & d_3 \\ -d_1 & -d_2 & -d_3 & d_4 \end{vmatrix}.$$

Skew determinants, in which the elements in the principal diagonal vanish, are called *skew symmetrical* determinants; for example

$$\begin{vmatrix} 0 & b_1 & c_1 & d_1 \\ -b_1 & 0 & c_2 & d_2 \\ -c_1 & -c_2 & 0 & d_3 \\ -d_1 & -d_2 & -d_3 & 0 \end{vmatrix}.$$

51. In the case of symmetrical determinants it is clear that the minors, corresponding to any two conjugate elements, are also equal. For the minors formed by omitting the p^{th} row and q^{th} column, and q^{th} row and p^{th} column respectively, differ only by an interchange of rows and columns.

For example in the symmetrical determinant

$$\begin{vmatrix} a & b & c \\ b & a & d \\ c & d & a \end{vmatrix}$$

the minor of b in the first row is $\begin{vmatrix} d & b \\ a & c \end{vmatrix}$, and of b in the first column $\begin{vmatrix} d & a \\ b & c \end{vmatrix}$, which is $= \begin{vmatrix} d & b \\ a & c \end{vmatrix}$.

Hence the reciprocal determinant of a symmetrical determinant is also symmetrical.

52. Again it is easily seen that in a skew symmetrical determinant the minor of any element differs from the minor of the conjugate element, by the sign of every element in the minor.

Hence if in a determinant of the n^{th} order the minor of any element be denoted by P, and the minor of the conjugate element by Q, we have $P = (-1)^{n-1}Q$. Thus $P = \pm Q$, according as n is odd or even.

For example in the determinant

$$\begin{vmatrix} 0 & b_1 & c_1 \\ -b_1 & 0 & c_2 \\ -c_1 & -c_2 & 0 \end{vmatrix},$$

the minor of b_1, *i.e.*, $\begin{vmatrix} c_2 & -b_1 \\ 0 & -c_1 \end{vmatrix} = \begin{vmatrix} -c_2 & 0 \\ b_1 & c_1 \end{vmatrix}$.

Hence the minors of b_1 and $-b_1$ are equal.

Again in the determinant

$$\begin{vmatrix} 0 & b_1 & c_1 & d_1 \\ -b_1 & 0 & c_2 & d_2 \\ -c_1 & -c_2 & 0 & d_3 \\ -d_1 & -d_2 & -d_3 & 0 \end{vmatrix}$$

the minor of b_1, *i.e.*, $\begin{vmatrix} c_2 & d_2 & -b_1 \\ 0 & d_3 & -c_1 \\ -d_3 & 0 & -d_1 \end{vmatrix} = -\begin{vmatrix} -c_2 & 0 & d_3 \\ -d_2 & -d_3 & 0 \\ b_1 & c_1 & d_1 \end{vmatrix}$,

that is, the minors of b_1 and $-b_1$ are equal in magnitude but differ in sign.

Hence the reciprocal determinant of a skew symmetrical determinant is a symmetrical, or skew symmetrical determinant, according as the given determinant is of odd or even degree.

53. Square of a Determinant.—If we multiply the determinant $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ by itself we obtain

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 = \begin{vmatrix} a_1^2 + b_1^2 & a_1a_2 + b_1b_2 \\ a_1a_2 + b_1b_2 & a_2^2 + b_2^2 \end{vmatrix}.$$

Similarly

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2 = \begin{vmatrix} a_1^2 + b_1^2 + c_1^2 & a_1a_2 + b_1b_2 + c_1c_2 & a_1a_3 + b_1b_3 + c_1c_3 \\ a_1a_2 + b_1b_2 + c_1c_2 & a_2^2 + b_2^2 + c_2^2 & a_2a_3 + b_2b_3 + c_2c_3 \\ a_1a_3 + b_1b_3 + c_1c_3 & a_2a_3 + b_2b_3 + c_2c_3 & a_3^2 + b_3^2 + c_3^2 \end{vmatrix}$$

and
$$\begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix}^2 = \begin{vmatrix} x^2 + y^2 + z^2 & xy + yz + zx & xy + yz + zx \\ xy + yz + zx & x^2 + y^2 + z^2 & xy + yz + zx \\ xy + yz + zx & xy + yz + zx & x^2 + y^2 + z^2 \end{vmatrix}.$$

From these examples we conclude that *the square of a determinant is a symmetrical determinant, and generally that every even power of a determinant is a symmetrical determinant.*

Again

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}^2 = \begin{vmatrix} 1 + 1 + 1 & a + b + c & a^2 + b^2 + c^2 \\ a + b + c & a^2 + b^2 + c^2 & a^3 + b^3 + c^3 \\ a^2 + b^2 + c^2 & a^3 + b^3 + c^3 & a^4 + b^4 + c^4 \end{vmatrix}$$

or

$$(a - b)^2(b - c)^2(c - a)^2 = \begin{vmatrix} a^0 + a^0 + a^0 & a + b + c & a^2 + b^2 + c^2 \\ a + b + c & a^2 + b^2 + c^2 & a^3 + b^3 + c^3 \\ a^2 + b^2 + c^2 & a^3 + b^3 + c^3 & a^4 + b^4 + c^4 \end{vmatrix},$$

a result of great importance in the Theory of Algebraic Equations.

It can easily be shown that the calculation of skew determinants reduces to that of skew symmetrical determinants; for example

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ -b_1 & b_2 & c_2 \\ -c_1 & -c_2 & c_3 \end{vmatrix} = \begin{vmatrix} 0 & b_1 & c_1 \\ -b_1 & 0 & c_2 \\ -c_1 & -c_2 & 0 \end{vmatrix}$$

$$+ a_1 \begin{vmatrix} 0 & c_2 \\ -c_2 & 0 \end{vmatrix} + b_2 \begin{vmatrix} 0 & c_1 \\ -c_1 & 0 \end{vmatrix} + c_3 \begin{vmatrix} 0 & b_1 \\ -b_1 & 0 \end{vmatrix} + a_1b_2c_3.$$

For a general proof of this theorem the student is referred to Salmon's Higher Algebra, Art. 41.

We need therefore only consider skew symmetrical determinants.

54. Theorem.—*A skew symmetrical determinant of an odd degree vanishes.*

Let $\triangle$ be a determinant of the n^{th} order. If we multiply each row of $\triangle$ by -1 we obtain

$$\triangle = (-1)^n \triangle \text{ (Art. 13)} = -\triangle \text{ if } n \text{ be odd,}$$

hence
$$\triangle + \triangle = 0$$

or
$$\triangle = 0.$$

Example.

$$\triangle = \begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix} = -\begin{vmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{vmatrix} = -\begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix} = -\triangle$$

whence
$$\triangle = 0.$$

55. Theorem.—*A skew symmetrical determinant of an even order is a complete square.*

For a determinant of the second order we get at once

$$\begin{vmatrix} 0 & a \\ -a & 0 \end{vmatrix} = a^2.$$

Consider next a determinant of the fourth order,

$$i.e., \begin{vmatrix} 0 & b_1 & c_1 & d_1 \\ -b_1 & 0 & c_2 & d_2 \\ -c_1 & -c_2 & 0 & d_3 \\ -d_1 & -d_2 & -d_3 & 0 \end{vmatrix} = \triangle.$$

By Art. 45 we have

$$\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = \triangle \begin{vmatrix} 0 & d_3 \\ -d_3 & 0 \end{vmatrix}$$

where A_1 and B_2 are the minors of the first and second zeros respectively in the dexter diagonal of Δ, and B_1 and A_2 the minors of b_1 and $-b_1$ respectively.

Now A_1 is a skew symmetrical determinant of the third order, and therefore vanishes (Art. 54), and $B_1 = -A_2$, since Δ is a determinant of the fourth order (Art. 52), and

hence $$(B_1)^2 = \Delta (d_3)^2,$$

therefore $$\Delta = \left(\frac{B_1}{d_3}\right)^2, \text{ a complete square.}$$

Similarly, if we have a determinant of the sixth order, we can show that $(B_1)^2 = \Delta \times \Delta'$, where Δ' is a skew symmetrical determinant of the fourth order, and is therefore by the preceding case a complete square, and thus Δ is itself a complete square.

In general, if Δ be a skew symmetrical determinant of the order $2n$, then $(B_1)^2 = \Delta \times \Delta'$, where Δ' is a skew symmetrical determinant of the order $2n - 2$. Thus Δ is a perfect square if Δ' is a perfect square.

Hence the theorem is true for determinants of order $2n$, if true for determinants of order $2n - 2$, but it is true for determinants of the second order, as we have already seen, and is therefore true for determinants of the fourth order, and therefore for determinants of the sixth order, and so on.

Examples.—

(1.) $$\Delta = \begin{vmatrix} 0 & a & b & c \\ -a & 0 & l & m \\ -b & -l & 0 & n \\ -c & -m & -n & 0 \end{vmatrix} = (bm - cl - an)^2.$$

For, by the above theorem, we have

$$\triangle = \left(\frac{B_1}{d_3}\right)^2 = \begin{vmatrix} -a & l & m \\ -b & 0 & n \\ -c & -n & 0 \end{vmatrix}^2 \div n^2$$

$$= \frac{(bmn - cln - an^2)^2}{n^2} = (bm - cl - an)^2.$$

(2.) $$\begin{vmatrix} 0 & a & b & 0 \\ -a & 0 & 0 & c \\ -b & 0 & 0 & d \\ 0 & -c & -d & 0 \end{vmatrix} = (ad - bc)^2.$$

(3.) $$\begin{vmatrix} 0 & a & 0 & b \\ -a & 0 & c & 0 \\ 0 & -c & 0 & d \\ -b & 0 & -d & 0 \end{vmatrix} = (ad + bc)^2.$$

(4.) $$\begin{vmatrix} 0 & a & -b & c \\ -a & 0 & 1 & 1 \\ b & -1 & 0 & 1 \\ -c & -1 & -1 & 0 \end{vmatrix} = (a + b + c)^2.$$

(5.) $$\begin{vmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & a & b \\ -1 & -a & 0 & c \\ -1 & -b & -c & 0 \end{vmatrix} = (a + b - c)^2.$$

(6.) $$\begin{vmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{vmatrix} = 1.$$

56. Additional Miscellaneous Examples.—

(1.)

$$\triangle = \begin{vmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{vmatrix} = (a^2 + b^2 + c^2 + d^2)^2.$$

Similarly

$$\triangle' = \begin{vmatrix} p & q & r & s \\ -q & p & -s & r \\ -r & s & p & -q \\ -s & -r & q & p \end{vmatrix} = (p^2+q^2+r^2+s^2)^2.$$

By multiplication we have $\triangle \times \triangle' =$
$(a^2+b^2+c^2+d^2)^2\,(p^2+q^2+r^2+s^2)^2 = (A^2+B^2+C^2+D^2)^2$
i.e., $(a^2+b^2+c^2+d^2)(p^2+q^2+r^2+s^2) = A^2+B^2+C^2+D^2$,
where $A = ap+bq+cr+ds$, $B = -aq+bp-cs+dr$,

$C = -ar+bs+cp-dq$, $D = -as-br+cq+dp$.

This furnishes another proof of Euler's theorem concerning the product of two numbers, each of which is the sum of four squares.

(2.) $\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = 4(b+c)\,(c+a)\,(a+b).$

(3.) $\begin{vmatrix} (b+c)^2 & c^2 & b^2 \\ c^2 & (c+a)^2 & a^2 \\ b^2 & a^2 & (a+b)^2 \end{vmatrix} = 2(bc+ca+ab)^3.$

(4.) $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3.$

(5.) $\begin{vmatrix} \dfrac{-bc}{b+c} & b & c \\ a & \dfrac{-ca}{c+a} & c \\ a & b & \dfrac{-ab}{a+b} \end{vmatrix} = \dfrac{(bc+ca+ab)^3}{(b+c)(c+a)(a+b)}.$

(6.) $\begin{vmatrix} (b+c)^2 & b^2 & c^2 \\ a^2 & (c+a)^2 & c^2 \\ a^2 & b^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3$

(7.) $\begin{vmatrix} 1 & a & -b \\ -a & 1 & c \\ b & -c & 1 \end{vmatrix} = 1+a^2+b^2+c^2.$

(8.) $$\begin{vmatrix} a & b & c & d \\ -a & b & m & n \\ -a & -b & c & g \\ -a & -b & -c & d \end{vmatrix} = 8abcd. \quad \text{(See Art. 32).}$$

(9.) $$\begin{vmatrix} 1 & a & a^2 & a^3 + bcd \\ 1 & b & b^2 & b^3 + cda \\ 1 & c & c^2 & c^3 + dab \\ 1 & d & d^2 & d^3 + abc \end{vmatrix} = 0.$$

(10.) $$\begin{vmatrix} 0 & a^2 & m^2 & d^2 \\ a^2 & 0 & b^2 & n^2 \\ m^2 & b^2 & 0 & c^2 \\ d^2 & n^2 & c^2 & 0 \end{vmatrix} = -(mn + ac + bd)(ac + bd - mn)(mn + ac - bd)(mn + bd - ac).$$

(11.) $$\begin{vmatrix} 0 & 1 & 1 & -1 \\ 1 & bc & -ac & ab \\ 1 & -bc & ac & ab \\ 1 & -bc & ac & -ab \end{vmatrix} = 4abc(a + b).$$

(12.) $$\begin{vmatrix} 0 & b & c \\ 2b^2c & 2bc & b^2 + c^2 - a^2 \\ 2bc^2 & b^2 + c^2 - a^2 & 2bc \end{vmatrix} = -4a^2b^2c^2.$$

(13.) $$\begin{vmatrix} 2xyz & z(x^2 + y^2) & y(x^2 + z^2) \\ z(x^2 + y^2) & 2xyz & x(y^2 + z^2) \\ y(z^2 + x^2) & x(y^2 + z^2) & 2xyz \end{vmatrix} = 0.$$

(14.) $$\begin{vmatrix} 1 & 1 & 1 & 1 \\ r & r_1 & 0 & 0 \\ r & 0 & r_2 & 0 \\ r & 0 & 0 & r_3 \end{vmatrix} = 0,$$

where r, r_1, r_2, and r_3 are the radii of the inscribed and three escribed circles of a triangle.

(15.) $$\begin{vmatrix} s-a & b & c & d \\ a & s-b & c & d \\ a & b & s-c & d \\ a & b & c & s-d \end{vmatrix} = 4bcd(s-a) + 4cda(s-b) + 4dab(s-c) + 4abc(s-d),$$

where $s = a + b + c + d$.

(16.) $$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1+a & 1+b & 1+c \\ 1 & a+1 & 0 & a+b & a+c \\ 1 & b+1 & b+a & 0 & b+c \\ 1 & c+1 & c+a & c+b & 0 \end{vmatrix}$$

$$= 8\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = 8abc\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

(17.) $$4b^2c^2\begin{vmatrix} a^2+b^2+c^2 & 2ab & 2ac \\ 2ab & a^2+b^2 & bc \\ 2ac & bc & a^2+c^2 \end{vmatrix}$$

$$= (b^2+c^2-a^2)^2\begin{vmatrix} b^2+c^2 & ab & ca \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix} = \{2abc(b^2+c^2-a^2)\}^2.$$

(18.) $$\begin{vmatrix} a & b & c-d \\ b & a-d & c \\ c-d & a & b \\ -d & c & b & a \end{vmatrix} = -16(s-a)(s-b)(s-c)(s-d),$$

where $s = \dfrac{a+b+c+d}{2}$.

(19.) $$\begin{vmatrix} -a(b^2+c^2-a^2) & 2b^3 & 2c^3 \\ 2a^3 & -b(c^2+a^2-b^2) & 2c^3 \\ 2a^3 & 2b^3 & -c(a^2+b^2-c^2 \end{vmatrix}$$
$$= abc(a^2+b^2+c^2)^3.$$

(20.) $$\begin{vmatrix} x & a_1 & a_2 & a_3 \\ -a_1 & x & b_2 & b_3 \\ -a_2 & -b_2 & x & c_3 \\ -a_3 & -b_3 & -c_3 & x \end{vmatrix} = x^4 + x^2(a_1^2 + a_2^2 + a_3^2 + b_2^2 + b_3^2 + c_3^2) + (a_1c_3 - a_2b_3 + a_3b_2)^2.$$

(21.) $$\begin{vmatrix} lx & my & nz \\ ax & by & cz \\ al & bm & cn \end{vmatrix} = \begin{vmatrix} bc & ca & ab \\ mn & nl & lm \\ yz & zx & xy \end{vmatrix}.$$

(22.) $$\begin{vmatrix} a+b+c+d & a-b-c+d & a-b+c-d \\ a-b-c+d & a+b+c+d & a+b-c-d \\ a-b+c-d & a+b-c-d & a+b+c+d \end{vmatrix}$$
$$= 16(bcd + cda + dab + abc).$$

(23.) $$\begin{vmatrix} x^2-yz & y^2-zx & z^2-xy \\ z^2-xy & x^2-yz & y^2-zx \\ y^2-zx & z^2-xy & x^2-yz \end{vmatrix} = \begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix}^2$$
$$= (x^3+y^3+z^3-3xyz).^2$$

(24.) $$\begin{vmatrix} 2a^2 & a^2+b^2-c^2 & c^2+a^2-b^2 \\ a^2+b^2-c^2 & 2b^2 & b^2+c^2-a^2 \\ c^2+a^2-b^2 & b^2+c^2-a^2 & 2c^2 \end{vmatrix}$$
$$= 4(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)$$
$$= 4\left\{a^4(b^2+c^2-a^2)+b^4(c^2+a^2-b^2)+c^4(a^2+b^2-c^2)-2a^2b^2c^2\right\}.$$

(25.) $$\begin{vmatrix} b^2+c^2-ab-ca & a^2+c^2-ab-bc & a^2+b^2-bc-ca \\ a^2+b^2-bc-ca & b^2+c^2-ab-ca & a^2+c^2-ab-bc \\ a^2+c^2-ab-bc & a^2+b^2-bc-ca & b^2+c^2-ab-ca \end{vmatrix}$$

$$= \frac{1}{8}\begin{vmatrix} b+c-a & a+c-b & a+b-c \\ a+b-c & b+c-a & a+c-b \\ a+c-b & a+b-c & b+c-a \end{vmatrix}^2 = 2\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}^2$$
$$= 2(a^3+b^3+c^3-3abc)^2.$$

(26.) $$\begin{vmatrix} 1 & 1 & 1 \\ bc(b+c) & ca(c+a) & ab(a+b) \\ b^2c^2 & c^2a^2 & a^2b^2 \end{vmatrix}$$
$$= abc(a+b+c)(a-b)(b-c)(a-c).$$

(27.) $$\begin{vmatrix} 1 & 1 & 1 \\ bc(c-b) & ca(a-c) & ab(b-a) \\ b^2c & c^2a & a^2b \end{vmatrix}$$
$$= abc(a^3+b^3+c^3-3abc).$$

(28.) $$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 2(s-b)(s-c) & sc & sb \\ 1 & sc & 2(s-c)(s-a) & sa \\ 1 & sb & sa & 2(s-a)(s-b) \end{vmatrix}$$
$$= -16s(s-a)(s-b)(s-c), \text{ where } 2s = a+b+c.$$

(29.) $$\begin{vmatrix} 3b-a & a+b & a+b & a+b \\ a+b & 3a-b & a+b & a+b \\ a+b & a+b & 3b-a & a+b \\ a+b & a+b & a+b & 3a-b \end{vmatrix} = \{2(a-b)\}^4.$$

(30.) $$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 6 & 12 & 20 \\ 3 & 12 & 30 & 60 \\ 4 & 20 & 60 & 140 \end{vmatrix} = 1 \times 2 \times 3 \times 4.$$

(31.) $$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 8 & 20 \\ 1 & 4 & 15 & 54 \\ 1 & 5 & 24 & 112 \end{vmatrix} = 1^3 \times 2^2 \times 3^1.$$

(32.) $$\begin{vmatrix} 4 & 3 & 2 & 1 \\ 36 & 63 & 81 & 90 \\ 16 & 44 & 80 & 120 \\ 12 & 45 & 105 & 195 \end{vmatrix} = 1^1 \times 2^2 \times 3^3 \times 4^4.$$

(33.) $$\begin{vmatrix} a & 0 & x & 0 \\ b & a & y & x \\ c & b & z & y \\ 0 & c & 0 & z \end{vmatrix} = (cx-az)^2 - (ay-bx)(bz-cy).$$

(34.) $$\begin{vmatrix} a & b & b & b \\ a & b & a & a \\ b & b & a & b \\ a & a & a & b \end{vmatrix} = -(a-b)^4.$$

(35.) $$\begin{vmatrix} 1 & 0 & a & x \\ 0 & 1 & b & y \\ 1 & 0 & c & z \\ 0 & 1 & d & w \end{vmatrix} = \begin{vmatrix} c-a & z-x \\ d-b & w-y \end{vmatrix}.$$

(36.) $$\begin{vmatrix} 0 & a^2 & b^2 & c^2 \\ a^2 & 2a^2 & a^2+b^2-c^2 & c^2+a^2-b^2 \\ b^2 & a^2+b^2-c^2 & 2b^2 & b^2+c^2-a^2 \\ c^2 & c^2+a^2-b^2 & b^2+c^2-a^2 & 2c^2 \end{vmatrix}$$

$$= -(a^2+b^2+c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)$$

$$= a^4(a^4-2b^4)+b^4(b^4-2c^4)+c^4(c^4-2a^4).$$

(37.) $$\begin{vmatrix} a & 2b & c & 0 \\ 0 & a & 2b & c \\ b & 2c & d & 0 \\ 0 & b & 2c & d \end{vmatrix} = (ad - bc)^2 - 4(b^2 - ac)(c^2 - bd).$$

(38.) $$\begin{vmatrix} 2a + b + c & c - a & b - a \\ c - b & 2b + c + a & a - b \\ b - c & a - c & 2c + a + b \end{vmatrix}$$

$$= 8(b + c)(c + a)(a + b).$$

(39.) $$\begin{vmatrix} 4a + b + c & c - b - 2a & b - c - 2a \\ c - a - 2b & 4b + c + a & a - c - 2b \\ b - a - 2c & a - b - 2c & 4c + a + b \end{vmatrix}$$

$$= 16(b + c)(c + a)(a + b).$$

(40.) $$\begin{vmatrix} 4a & 3a - b - c & 3a - b - c \\ 3b - c - a & 4b & 3b - c - a \\ 3c - a - b & 3c - a - b & 4c \end{vmatrix} = 2(a + b + c)^3.$$

(41.) $$\begin{vmatrix} a^2 & bc & bd & -ca \\ bc & a^2 & ab & -cd \\ -ca & cd & d^2 & c^2 \\ -bd & ab & b^2 & d^2 \end{vmatrix} = (a^2d^2 - b^2c^2)^2.$$

(42.) $$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & a^2 + \alpha^2 & ab + \alpha\beta & ac + \alpha\gamma \\ 1 & ab + \alpha\beta & b^2 + \beta^2 & bc + \beta\gamma \\ 1 & ac + \alpha\gamma & bc + \beta\gamma & c^2 + \gamma^2 \end{vmatrix}$$

$$= -\{\alpha(b - c) + \beta(c - a) + \gamma(a - b)\}^2.$$

(43.) $$\begin{vmatrix} x_1 & a & a & 1 \\ b & x_2 & a & 1 \\ b & b & x_3 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$$

$$= \frac{(x_1 - a)(x_2 - a)(x_3 - a) - (x_1 - b)(x_2 - b)(x_3 - b)}{a - b}.$$

(44.) $$\begin{vmatrix} 0 & a & a & 1 \\ b & 0 & a & 1 \\ b & b & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = -\left(a^2 + ab + b^2\right).$$

(45.) $$\begin{vmatrix} x_1 & a & a & a \\ b & x_2 & a & a \\ b & b & x_3 & a \\ b & b & b & x_4 \end{vmatrix}$$

$$= \frac{a(x_1 - b)(x_2 - b)(x_3 - b)(x_4 - b) - b(x_1 - a)(x_2 - a)(x_3 - a)(x_4 - a)}{a - b}.$$

Ex. 20, Art. 29 is a particular case of this.

(46.) $$\begin{vmatrix} 3a & a & a & a \\ 2a - b & 2a + b & b & b \\ a & a & 2b + a & 2b - a \\ b & b & b & 3b \end{vmatrix} = 16ab\left(a^2 + ab + b^2\right).$$

(47.) $$\begin{vmatrix} b + c & b - c & c - b \\ a - c & c + a & c - a \\ a - b & b - a & a + b \end{vmatrix} = 8abc.$$

(48.) $$\begin{vmatrix} 2a + 1 & 1 & 1 & 2a - 1 \\ a - b + 1 & a + b + 1 & b - a + 1 & b + a - 1 \\ 1 & 1 & 2b + 1 & 2b - 1 \\ 1 & 1 & 1 & 3 \end{vmatrix}$$

$$= 16(a^2 + ab + b^2).$$

(49.) $$\begin{vmatrix} 1 & 1 & 1 \\ a - b & b - c & c - a \\ (a - b)^2 & (b - c)^2 & (c - a)^2 \end{vmatrix}$$

$$= -(2a - b - c)(2b - c - a)(2c - a - b).$$

(50.) $$\begin{vmatrix} a + b & b + c & c + a \\ a + b - 1 & b + c - 1 & c + a - 1 \\ (a + b)(a + b - 1) & (b + c)(b + c - 1) & (c + a)(c + a - 1) \end{vmatrix}$$

$$= (a - b)(b - c)(a - c).$$

(51.) $\begin{vmatrix} a+b & c+a & b+c \\ b+c & a+b & c+a \\ c+a & b+c & a+b \end{vmatrix} = 2(a^3+b^3+c^3-3abc).$

(52.) $\begin{vmatrix} ab+bc & ca+ab & bc+ca \\ bc+ca & ab+bc & ca+ab \\ ca+ab & bc+ca & ab+bc \end{vmatrix} = 2\begin{vmatrix} ab & bc & ca \\ ca & ab & bc \\ bc & ca & ab \end{vmatrix}$

$$= 2(a^3b^3+b^3c^3+c^3a^3-3a^2b^2c^2).$$

(53). $\begin{vmatrix} 0 & 1 & \alpha & \alpha^2 \\ -1 & 0 & -\alpha^2 & \alpha \\ -\alpha & \alpha^2 & 0 & -1 \\ -\alpha^2 & \alpha & 1 & 0 \end{vmatrix} = 0,$ where $\alpha^3 = 1$.

(54). $\begin{vmatrix} 1 & 1 & 1 & 1 \\ abc & bcd & cda & dab \\ a^2b^2c^2 & b^2c^2d^2 & c^2d^2a^2 & d^2a^2b^2 \\ a^3b^3c^3 & b^3c^3d^3 & c^3d^3a^3 & d^3a^3b^3 \end{vmatrix}$

$$= a^3b^3c^3d^3(a-b)(a-c)(a-d)(b-c)(b-d)(d-c).$$

(55.) $\begin{vmatrix} 0 & x_1-x_2 & x_1-x_3 & x_1-x_4 \\ x_2-x_1 & 0 & x_2-x_3 & x_2-x_4 \\ x_3-x_1 & x_3-x_2 & 0 & x_3-x_4 \\ x_4-x_1 & x_4-x_2 & x_4-x_3 & 0 \end{vmatrix} = 0.$

(56). $\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ bcd & cda & abd & abc \\ b^2c^2d^2 & c^2d^2a^2 & a^2b^2d^2 & a^2b^2c^2 \end{vmatrix}$

$$= abcd(a-b)(a-c)(a-d)(b-c)(b-d)(d-c).$$

(57). $\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & a+b & a+c \\ 1 & b+a & 0 & b+c \\ 1 & c+a & c+b & 0 \end{vmatrix} = abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$

CHAPTER V.

APPLICATIONS OF DETERMINANTS.

57. Solution of Equations expressed as Determinants. —As this part of the subject is not an important one, we will merely give one or two examples as illustrations, adding a number for practice.

For example, suppose we have to solve the determinantal equation,

(1.)
$$\Delta = \begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix} = 0.$$

By reducing the determinant we obtain

$$\Delta = (x-1)^2(x+2) = 0,$$

whence $x = 1$ and $x = -2$.

(2.)
$$\Delta = \begin{vmatrix} x-a & x-b & x-c \\ x-c & x-a & x-b \\ x-b & x-c & x-a \end{vmatrix} = 0.$$

We have (See ex. 9, Art. 29)

$$\Delta = (x-a)^3 + (x-b)^3 + (x-c)^3 - 3(x-a)(x-b)(x-c) = 0,$$

or $(\overline{x-a} + \overline{x-b} + \overline{x-c})\ \{(x-a)^2 + (x-b)^2 + (x-c)^2 -$
$(x-a)(x-b) - (x-b)(x-c) - (x-c)(x-a)\} = 0,$

whence $(\overline{x-a} + \overline{x-b} + \overline{x-c}) = 0$,

therefore $x = \dfrac{a+b+c}{3}$.

(3.)
$$\Delta = \begin{vmatrix} x & b & c & d \\ b & x & d & c \\ c & d & x & b \\ d & c & b & x \end{vmatrix} = 0.$$

We have (Ex. 18, Art 48)

$$\Delta = -(x+b+c+d)(x+b-c-d)(x+d-b-c)(-x+d+b-c) = 0$$

hence the four values of x are

$$-b-c-d,\ c+d-b,\ b+c-d, \text{ and } d+b-c.$$

Additional Examples :

(4).
$$\begin{vmatrix} x & a & a & \dots & a \\ a & x & a & \dots & a \\ a & a & x & \dots & a \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ a & a & a & \dots & x \end{vmatrix} \text{(n rows)} = 0, \ \{x = a, \text{ and } a(1-n)\}.$$

(5.)
$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ x & a & 0 & 0 \\ x & 0 & b & 0 \\ x & 0 & 0 & c \end{vmatrix} = 0. \quad \left(x = \frac{abc}{ab+bc+ca}.\right)$$

(6.)
$$\begin{vmatrix} x & a & b & c \\ a & x & 0 & 0 \\ b & 0 & x & 0 \\ c & 0 & 0 & x \end{vmatrix} = 0. \quad (x = 0 \text{ and } \pm\sqrt{a^2+b^2+c^2}).$$

(7.) $$\begin{vmatrix} 1 & x & x^2 \\ 1 & a & a^2 \\ 1 & b & b^2 \end{vmatrix} = 0. \quad (x = a, \text{ and } = b).$$

(8.) $$\begin{vmatrix} x-b-c & 2x & 2x \\ 2b & b-c-x & 2b \\ 2c & 2c & c-b-x \end{vmatrix} = 0. \quad \{x = -(b+c)\}.$$

(9.) $$\begin{vmatrix} 0 & a^2 & x^2 & d^2 \\ a^2 & 0 & b^2 & x^2 \\ x & b^2 & 0 & c^2 \\ d^2 & x^2 & c^2 & 0 \end{vmatrix} = 0.$$

$$\left\{ x = \begin{matrix} \pm\sqrt{-(ac+bd)}, & \pm\sqrt{(ac+bd)} \\ \pm\sqrt{bd-ac}, & \pm\sqrt{ac-bd} \end{matrix} \right\}.$$

Reduce the following determinantal equations to the solution of a cubic and a quadratic equation respectively:

(10.) $$\begin{vmatrix} a & x & x \\ x & b & x \\ x & x & c \end{vmatrix} = 0.$$

(11.) $$\begin{vmatrix} a^3 & b^3 & c^3 \\ (a+x)^3 & (b+x)^3 & (c+x)^3 \\ (2a+x)^3 & (2b+x)^3 & (2c+x)^3 \end{vmatrix} = 0.$$

58. Solution of a system of n Equations of the first degree between n Variables.—This is one of the most important of the applications of determinants, but we will only give one or two of the more elementary propositions on the subject.

To make our demonstrations as simple as possible we will consider three equations between three variables, but the method applies equally in general.

Let the three equations be

$$\begin{cases} a_1x + b_1y + c_1z = d_1, \\ a_2x + b_2y + c_2z = d_2, \\ a_3x + b_3y + c_3z = d_3. \end{cases}$$

Since $a_1x + b_1y + c_1z - d_1 = 0$,

$a_2x + b_2y + c_2z - d_2 = 0$,

and $a_3x + b_3y + c_3z - d_3 = 0$;

we have at once

$$\begin{vmatrix} a_1x + b_1y + c_1z - d_1 & b_1 & c_1 \\ a_2x + b_2y + c_2z - d_2 & b_2 & c_2 \\ a_3x + b_3y + c_3z - d_3 & b_3 & c_3 \end{vmatrix} = 0.$$

This determinant can be resolved into the sum of four determinants, two of which vanish (Cor. III. Art. 13).

The two that remain are

$$\begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix} = x\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = x\Delta,$$

and
$$\begin{vmatrix} -d_1 & b_1 & c_1 \\ -d_2 & b_2 & c_2 \\ -d_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = -\Delta'.$$

Hence $x\Delta - \Delta' = 0$

and therefore $x = \dfrac{\Delta'}{\Delta} = \dfrac{d_1A_1 + d_2A_2 + d_3A_3}{a_1A_1 + a_2A_2 + a_3A_3}.$

In a similar manner it is found that

$$y = \frac{\Delta''}{\Delta} = \frac{d_1B_1 + d_2B_2 + d_3B_3}{b_1B_1 + b_2B_2 + b_3B_3},$$

and $z = \dfrac{\Delta'''}{\Delta} = \dfrac{d_1C_1 + d_2C_2 + d_3C_3}{c_1C_1 + c_2C_2 + c_3C_3}.$

The above result may be obtained more simply thus:

Multiply the first equation by A_1, the second by A_2, and the third by A_3, when we get at once

$$\triangle x = d_1A_1 + d_2A_2 + d_3A_3.$$

Similarly by multiplying by B_1, B_2, B_3 and C_1, C_2, C_3 we have

$$\triangle y = d_1B_1 + d_2B_2 + d_3B_3$$
$$\triangle z = d_1C_1 + d_2C_2 + d_3C_3.$$

Thus each variable can be expressed as a fraction, whose denominator is the determinant, whose elements are the co-efficients of all the variables, and whose numerator is the same determinant with the co-efficients of the variable, whose value we wish to determine, replaced by the constant terms of the equations.

We will now prove that the values of x, y, and z, found as above, satisfy the given equations. For substituting, in the first equation, for example, these values of x, y, and z, we obtain

$$\frac{a_1(d_1A_1 + d_2A_2 + d_3A_3)}{\triangle} + \frac{b_1(d_1B_1 + d_2B_2 + d_3B_3)}{\triangle} + \frac{c_1(d_1C_1 + d_2C_2 + d_3C_3)}{\triangle} = d_1,$$

or $d_1(a_1A_1 + b_1B_1 + c_1C_1) + d_2(a_1A_2 + b_1B_2 + c_1C_2) + d_3(a_1A_3 + b_1B_3 + c_1C_3) = d_1\triangle.$

Now $a_1A_1 + b_1B_1 + c_1C_1 = \triangle,$
$a_1A_2 + b_1B_2 + c_1C_2 = 0,$
and $a_1A_3 + b_1B_3 + c_1C_3 = 0;$

hence $d_1\triangle = d_1\triangle.$

In a similar manner it can be shown that the same values of x, y, and z satisfy the other two equations.

59. If all the constant terms of the equations vanish except one, for example, if the given equations be

$$\begin{cases} a_1x + b_1y + c_1z = d_1, \\ a_2x + b_2y + c_2z = 0, \\ a_3x + b_3y + c_3z = 0, \end{cases}$$

then we have $x = \dfrac{d_1A_1}{\Delta}$, $y = \dfrac{d_1B_1}{\Delta}$ and $z = \dfrac{d_1C_1}{\Delta}$

and therefore $\dfrac{x}{A_1} = \dfrac{y}{B_1} = \dfrac{z}{C_1} = \dfrac{d_1}{\Delta}$.

From this we conclude that in a system of n equations of the first degree between n variables, and in which all the constant terms but one vanish, the values of the variables are proportional to the minors of the co-efficients of the variables in the equation, whose constant term does not vanish.

60. If now all the constant terms vanish; for example, if the given system of Equations be

$$\begin{cases} a_1x + b_1y + c_1z = 0, \\ a_2x + b_2y + c_2z = 0, \\ a_3x + b_3y + c_3z = 0, \end{cases}$$

then $x = \dfrac{0}{\Delta}$, $y = \dfrac{0}{\Delta}$ and $z = \dfrac{0}{\Delta}$;

hence if Δ be not $= 0$, then $x = y = z = 0$.

In order therefore that three such equations may be compatible, that is, admit of solutions which are not zero, we must have

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

If this relation holds amongst the co-efficients the equations may be satisfied by

$$\frac{x}{A_1} = \frac{y}{B_1} = \frac{z}{C_1},$$

or $$\frac{x}{A_2} = \frac{y}{B_2} = \frac{z}{C_2},$$

or $$\frac{x}{A_3} = \frac{y}{B_3} = \frac{z}{C_3}.$$

Hence we conclude that in a system of n equations between n variables, and in which all the constant terms vanish, the values of the variables are proportional to the determinant Minors of the co-efficients of the variables in any of the equations.

We have also the following relations between the Minors:

$$\frac{A_1}{B_1} = \frac{A_2}{B_2} = \frac{A_3}{B_3},$$

$$\frac{B_1}{C_1} = \frac{B_2}{C_2} = \frac{B_3}{C_3},$$

$$\frac{C_1}{A_1} = \frac{C_2}{A_2} = \frac{C_3}{A_3}.$$

See also Corollary, Art. 45.

61. Examples:—

(1). Solve

$$\begin{cases} 5x + 3y + 3z = 48, \\ 2x + 6y - 3z = 18, \\ 8x - 3y + 2z = 21 \end{cases}$$

We have by the rule

$$x = \begin{vmatrix} 48 & 3 & 3 \\ 18 & 6 & -3 \\ 21 & -3 & 2 \end{vmatrix} \div \begin{vmatrix} 5 & 3 & 3 \\ 2 & 6 & -3 \\ 8 & -3 & 2 \end{vmatrix}$$

$$= -693 \div -231 = 3.$$

Similarly $y = 5$, and $z = 6$.

(2.) $\begin{cases} 4y - 3z = 1 \\ 3x - 2z = 8 \\ 5x - 7y = 2 \end{cases}$

$$x = \begin{vmatrix} 1 & 4 & -3 \\ 8 & 0 & -2 \\ 2 & -7 & 0 \end{vmatrix} \div \begin{vmatrix} 0 & 4 & -3 \\ 3 & 0 & -2 \\ 5 & -7 & 0 \end{vmatrix}$$

$$= 138 \div 23 = 6.$$

Similarly $y = 4$, and $z = 5$.

(3.) $\begin{cases} x - ay + a^2z = a^3 \\ x - by + b^2z = b^3 \\ x - cy + c^2z = c^3 \end{cases}$

We have

$$x = \begin{vmatrix} a^3 & -a & a^2 \\ b^3 & -b & b^2 \\ c^3 & -c & c^2 \end{vmatrix} \div \begin{vmatrix} 1 & -a & a^2 \\ 1 & -b & b^2 \\ 1 & -c & c^2 \end{vmatrix}$$

$$= abc \begin{vmatrix} a^2 & -1 & a \\ b^2 & -1 & b \\ c^2 & -1 & c \end{vmatrix} \div \begin{vmatrix} 1 & -a & a^2 \\ 1 & -b & b^2 \\ 1 & -c & c^2 \end{vmatrix}$$

$$= abc.$$

Similarly $y = ab + bc + ca$ and $z = a + b + c$.

(4.) $$\left\{\begin{array}{c} x + y + z = 0 \\ (b + c)x + (c + a)y + (a + b)z = 0 \\ bcx + cay + abz = 1 \end{array}\right\} \begin{array}{l} x = \dfrac{1}{(a - b)(a - c)} \\ y = \dfrac{1}{(b - c)(b - a)} \\ z = \dfrac{1}{(c - a)(c - b)}. \end{array}$$

(5.) $$\left\{\begin{array}{l} ax + by + cz + du = 1 \\ bx + cy + dz + au = 1 \\ cx + dy + az + bu = 1 \\ dx + ay + bz + cu = 1 \end{array}\right\} x = y = z = u = \frac{1}{a + b + c + d}.$$

(6.) $$\left\{\begin{array}{l} y + z + u = a \\ z + u + x = b \\ u + x + y = c \\ x + y + z = d \end{array}\right\}. \begin{array}{l} x = \dfrac{b + c + d - 2a}{3} \\ y = \dfrac{a + c + d - 2b}{3} \end{array}$$

&c.

(7.) $$\left\{\begin{array}{c} ax + by + cz = d \\ a^2x + b^2y + c^2z = d^2 \\ a^3x + b^3y + c^3z = d^3 \end{array}\right\}. \quad x = \frac{d(d - b)(d - c)}{a(a - b)(a - c)}.$$

&c.

(8.) $$\left\{\begin{array}{l} bx + ay = c \\ cx + az = b \\ cy + bz = a \end{array}\right\}. \quad x = \frac{b^2 + c^2 - a^2}{2bc}$$

&c.

(9.) $$\left\{\begin{array}{l} \dfrac{x}{a} + \dfrac{y}{b} = 1 \\ \dfrac{x}{a} + \dfrac{z}{c} = 1 \\ \dfrac{y}{b} + \dfrac{z}{c} = 1 \end{array}\right\} \begin{array}{l} x = \dfrac{a}{2} \\ y = \dfrac{b}{2} \\ z = \dfrac{c}{2}. \end{array}$$

(10.) $$\left.\begin{cases} \dfrac{a}{x} + \dfrac{b}{y} = 1 \\ \dfrac{b}{y} + \dfrac{c}{z} = 1 \\ \dfrac{c}{z} + \dfrac{a}{x} = 1 \end{cases}\right\} \cdot \quad \begin{matrix} x = 2a. \\ y = 2b. \\ z = 2c. \end{matrix}$$

62. Condition that n Linear Equations between $n - 1$ Variables may be consistent.—Let us consider the four equations.

$$\begin{cases} a_1x + b_1y + c_1z + d_1 = 0. \\ a_2x + b_2y + c_2z + d_2 = 0, \\ a_3x + b_3y + c_3z + d_3 = 0, \\ a_4x + b_4y + c_4z + d_4 = 0. \end{cases}$$

We have already seen that, in order that the system

$$\begin{cases} a_1x + b_1y + c_1z + d_1w = 0, \\ a_2x + b_2y + c_2z + d_2w = 0, \\ a_3x + b_3y + c_3z + d_3w = 0, \\ a_4x + b_4y + c_4z + d_4w = 0, \end{cases}$$

may be consistent, we must have

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0.$$

Hence, since the given system of equations can be deduced from this system by putting $w = 1$, we see that, in order that the given system may be consistent, *the determinant whose elements are all the co-efficients of the variables and the constant terms must vanish.*

This important proposition can also be proved in another way.

In order that the fourth equation may be consistent with the other three, the values of x, y, z, found from the first three, must satisfy the fourth.

The condition for consistency is therefore

$$a_4\frac{\Delta'}{\Delta} + b_4\frac{\Delta''}{\Delta} + c_4\frac{\Delta'''}{\Delta} + d_4 = 0$$

or $$a_4\Delta' + b_4\Delta'' + c_4\Delta''' + d_4\Delta = 0$$

that is $a_4A_4 + b_4B_4 + c_4C_4 + d_4D_4 = 0$

i.e. $$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0 \text{ as before.}$$

Examples:

The equations

$$\begin{aligned} ax + by + c &= 0 \\ cx + ay + b &= 0 \\ bx + cy + a &= 0 \end{aligned}$$

are consistent if

$$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = 0, \text{ that is, if } a^3 + b^3 + c^3 - 3abc = 0.$$

Similarly the equations

$$\begin{cases} (b+c)x + (c+a)y + (a+b) = 0 \\ (c+a)x + (a+b)y + (b+c) = 0 \\ (a+b)x + (b+c)y + (c+a) = 0 \end{cases}$$

are consistent if

$$\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0 \text{ i.e. } a^3 + b^3 + c^3 - 3abc = 0.$$

63. We are now in a position to give another demonstration of the Theorem in Art. 45.

For let us consider the system of equations

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1, \\ a_2x + b_2y + c_2z &= d_2, \\ a_3x + b_3y + c_3z &= d_3. \end{aligned}$$

Solving for x, y and z we obtain,

$$\begin{aligned} \triangle \times x &= d_1A_1 + d_2A_2 + d_3A_3, \\ \triangle \times y &= d_1B_1 + d_2B_2 + d_3B_3, \\ \triangle \times z &= d_1C_1 + d_2C_2 + d_3C_3. \quad \text{(Art. 58).} \end{aligned}$$

If we now solve for d_1, d_2 and d_3 in this second system of equations, we have

$$\begin{aligned} \triangle' \times d_1 &= \alpha_1 \triangle x + \alpha_2 \triangle y + \alpha_3 \triangle z, \\ \triangle' \times d_2 &= \beta_1 \triangle x + \beta_2 \triangle y + \beta_3 \triangle z, \\ \triangle' \times d_3 &= \gamma_1 \triangle x + \gamma_2 \triangle y + \gamma_3 \triangle z\,; \end{aligned}$$

where $\triangle' = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \triangle^2$ (Art. 44),

and $\alpha_1 = \begin{vmatrix} B_2 & C_2 \\ B_3 & C_3 \end{vmatrix}$, $\alpha_2 = \begin{vmatrix} B_3 & C_3 \\ B_1 & C_1 \end{vmatrix}$ &c.

Hence we have

$$\triangle^2 \times d_1 = \alpha_1 \triangle x + \alpha_2 \triangle y + \alpha_3 \triangle z,$$

&c.

or $$d_1 = \alpha_1 \triangle^{1-2} x + \alpha_2 \triangle^{1-2} y + \alpha_3 \triangle^{1-2} z$$

&c.

But as above

$$d_1 = a_1x + b_1y + c_1z,$$

and therefore

$$a_1x + b_1y + c_1z = \alpha_1 \Delta^{1-2}x + \alpha_2 \Delta^{1-2}y + \alpha_3 \Delta^{1-2}z,$$

or

$$a_1 = \alpha_1 \Delta^{1-2},$$

that is

$$a_1 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} B_2 & C_2 \\ B_3 & C_3 \end{vmatrix}.$$

We will next prove as in Art. 45 that

$$\begin{vmatrix} C_3 & D_3 \\ C_4 & D_4 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}.$$

Solving for z and w in the system of equations

$$\begin{aligned} a_1x + b_1y + c_1z + d_1w &= e_1, \\ a_2x + b_2y + c_2z + d_2w &= e_2, \\ a_3x + b_3y + c_3z + d_3w &= e_3, \\ a_4x + b_4y + c_4z + d_4w &= e_4, \end{aligned}$$

we obtain

$$\begin{aligned} \Delta z &= e_1 C_1 + e_2 C_2 + e_3 C_3 + e_4 C_4, \\ \Delta w &= e_1 D_1 + e_2 D_2 + e_3 D_3 + e_4 D_4. \end{aligned}$$

Hence multiplying the first of these equations by D_4 and the second by C_4, and subtracting, we have

$$D_4 \Delta z - C_4 \Delta w = e_1(D_4C_1 - D_1C_4) + e_2(D_4C_2 - D_2C_4) + e_3(D_4C_3 - D_3C_4)$$

$$= e_1\begin{vmatrix} C_1 & C_4 \\ D_1 & D_4 \end{vmatrix} + e_2\begin{vmatrix} C_2 & C_4 \\ D_2 & D_4 \end{vmatrix} + e_3\begin{vmatrix} C_3 & C_4 \\ D_3 & D_4 \end{vmatrix}.$$

But if we eliminate x and y from the first three equations of the original system, we obtain

$$z\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_2 \end{vmatrix} + w\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} = \begin{vmatrix} e_1 - d_1w & a_1 & b_1 \\ e_2 - d_2w & a_2 & b_2 \\ e_3 - d_3w & a_3 & b_3 \end{vmatrix} + \begin{vmatrix} e_1 - c_1z & a_1 & b_1 \\ e_2 - c_2z & a_2 & b_2 \\ e_3 - c_3z & a_3 & b_3 \end{vmatrix}$$

or

$$z\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + w\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} = \begin{vmatrix} e_1 & a_1 & b_1 \\ e_2 & a_2 & b_2 \\ e_3 & a_3 & b_3 \end{vmatrix}$$

that is

$$D_4z - C_4w = e_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} + e_2\begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix} + e_3\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

Hence comparing this result with the above, we find

$$\begin{vmatrix} C_3 & D_3 \\ C_4 & D_4 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\Delta, \text{ \&c.}$$

The method in which the general theorem is proved will be sufficiently obvious from the proofs in the preceding cases.

We can also give another proof that the product of two determinants is a determinant.

For example, let us consider two systems of three equations, such as

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\},$$

$$\text{and}\quad \left.\begin{aligned} \alpha_1 d_1 + \beta_1 d_2 + \gamma_1 d_3 &= 0 \\ \alpha_2 d_1 + \beta_2 d_2 + \gamma_2 d_3 &= 0 \\ \alpha_3 d_1 + \beta_3 d_2 + \gamma_3 d_3 &= 0 \end{aligned}\right\}.$$

If we now substitute in the second system the values of d_1, d_2 and d_3 obtained from the first, and collect the coefficients of x, y and z, we obtain

$$\left.\begin{aligned} (a_1\alpha_1 + a_2\beta_1 + a_3\gamma_1)x + (b_1\alpha_1 + b_2\beta_1 + b_3\gamma_1)y + (c_1\alpha_1 + c_2\beta_1 + c_3\gamma_1)z &= 0 \\ (a_1\alpha_2 + a_2\beta_2 + a_3\gamma_2)x + (b_1\alpha_2 + b_2\beta_2 + b_3\gamma_2)y + (c_1\alpha_2 + c_2\beta_2 + c_3\gamma_2)z &= 0 \\ (a_1\alpha_3 + a_2\beta_3 + a_3\gamma_3)x + (b_1\alpha_3 + b_2\beta_3 + b_3\gamma_3)y + (c_1\alpha_3 + c_2\beta_3 + c_3\gamma_3)z &= 0 \end{aligned}\right\}.$$

Now the condition of consistency of this third system of equations is

$$\Delta = \begin{vmatrix} a_1\alpha_1 + a_2\beta_1 + a_3\gamma_1 & b_1\alpha_1 + b_2\beta_1 + b_3\gamma_1 & c_1\alpha_1 + c_2\beta_1 + c_3\gamma_1 \\ a_1\alpha_2 + a_2\beta_2 + a_3\gamma_2 & b_1\alpha_2 + b_2\beta_2 + b_3\gamma_2 & c_1\alpha_2 + c_2\beta_2 + c_3\gamma_2 \\ a_1\alpha_3 + a_2\beta_3 + a_3\gamma_3 & b_1\alpha_3 + b_2\beta_3 + b_3\gamma_3 & c_1\alpha_3 + c_2\beta_3 + c_3\gamma_3 \end{vmatrix} = 0 \text{ (Art. 60).}$$

But if we write the original equations thus:

$$\begin{aligned} a_1x + b_1y + c_1z - d_1 \qquad\qquad\qquad\qquad &= 0, \\ a_2x + b_2y + c_2z \qquad - d_2 \qquad\qquad &= 0, \\ a_3x + b_3y + c_3z \qquad\qquad - d_3 &= 0, \\ + \alpha_1 d_1 + \beta_1 d_2 + \gamma_1 d_3 &= 0, \\ + \alpha_2 d_1 + \beta_2 d_2 + \gamma_2 d_3 &= 0, \\ + \alpha_3 d_1 + \beta_3 d_2 + \gamma_2 d_3 &= 0, \end{aligned}$$

the condition of consistency is at once

$$\Delta' = \begin{vmatrix} a_1 & b_1 & c_1 & -1 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & -1 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 & -1 \\ 0 & 0 & 0 & \alpha_1 & \beta_1 & \gamma_1 \\ 0 & 0 & 0 & \alpha_2 & \beta_2 & \gamma_2 \\ 0 & 0 & 0 & \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = 0 \text{ (Art. 60)}$$

Now

$$\Delta' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} \text{ (Art 35).}$$

This must be the same condition as that already found, and therefore Δ and Δ' can only differ by a numerical factor which is seen to be unity on equating the co-efficients of any term say $a_1\, b_2\, c_3\, \alpha_1\, \beta_2\, \gamma_3$.

This proof has been taken with some slight modifications from the Chapter on Determinants in the "Messenger of Mathematics" by Professor P. G. Tait.

64. Trigonometrical Applications.—We can also express in determinant notation several trigonometrical relations.

For example; we have (Todhunter's Trigonometry, Art. 216),

$$\begin{aligned} a &= c \cos B + b \cos C \\ b &= a \cos C + c \cos A \\ c &= b \cos A + a \cos B, \end{aligned} \quad \text{that is} \quad \begin{aligned} -a + b\cos C + c\cos B &= 0 \\ a\cos C - b + c\cos A &= 0 \\ a\cos B + b\cos A - c &= 0. \end{aligned}$$

The condition that these equations may be consistent is

$$\begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix} = 0,$$

which gives

$$\mathrm{Cos}^2 A + \cos^2 B + \cos^2 C + 2 \cos A . \cos B . \cos C = 1.$$

Again

$$a^2 = b^2 + c^2 - 2bc \cos A = \begin{vmatrix} b & 0 & c \\ 1 & b & \cos A \\ \cos A & c & 1 \end{vmatrix}.$$

We add three trigonometrical identities for verification by the student.

(1.) $$\begin{vmatrix} 1 & 1 & 1 \\ \sin A & \sin B & \sin C \\ \cos A & \cos B & \cos C \end{vmatrix}$$

$$= \sin(B - C) + \sin(C - A) + \sin(A - B)$$

also $= 4 \sin \frac{1}{2}(A - B). \sin \frac{1}{2}(B - C). \sin \frac{1}{2}(A - C).$

(2.) $$\begin{vmatrix} \sin A & \sin B & \sin C \\ \cos A & \cos B & \cos C \\ \sin A. \cos A & \sin B. \cos B & \sin C. \cos C \end{vmatrix}$$

$$= 2 \sin \tfrac{1}{2}(A - B) \sin \tfrac{1}{2}(B - C) \sin \tfrac{1}{2}(C - A)$$
$$\left\{\sin(A + B) + \sin(B + C) + \sin(C + A)\right\}$$
$$= -\tfrac{1}{2}\left\{\sin(A - B) + \sin(B - C) + \sin(C - A)\right\}$$
$$\left\{\sin(A + B) + \sin(B + C) + \sin(C + A)\right\}.$$

(3.) $$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & \cos C & 1 & \cos B \\ 1 & 1 & \cos C & \cos A \\ 1 & \cos A & \cos B & 1 \end{vmatrix} = 16 \sin^2 \frac{A}{2} \sin^2 \frac{B}{2} \sin^2 \frac{C}{2}.$$

65. Relations between the roots and co-efficients of a cubic equation.

Let the equation be $x^3 + px^2 + qx + r = 0$,

and let the roots be a, b, and c.

We get at once
$$a^3 + pa^2 + qa + r = 0,$$
$$b^3 + pb^2 + qb + r = 0,$$
$$c^3 + pc^2 + qc + r = 0.$$

Solving for p we have

$$p = -\begin{vmatrix} a^3 & a & 1 \\ b^3 & b & 1 \\ c^3 & c & 1 \end{vmatrix} \div \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}$$

$$= -(a+b+c)\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} \div \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}$$

$$= -(a+b+c).$$

Similarly $q = \begin{vmatrix} a^3 & a^2 & 1 \\ b^3 & b^2 & 1 \\ c^3 & c^2 & 1 \end{vmatrix} \div \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = (ab+bc+ca),$

and $r = -\begin{vmatrix} a^3 & a^2 & a \\ b^3 & b^2 & b \\ c^3 & c^2 & c \end{vmatrix} \div \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = -abc.$

This method might be extended to equations of higher degree.

66. Solution of a Cubic Equation.—We conclude by showing how we can solve a cubic equation by means of determinants.

We have seen (Cor. ex. 17, Art. 48) that

$$a^3+b^3+c^3-3abc = (a+b+c)(a+\omega^2 b+\omega c)(a+\omega b+\omega^2 c),$$

where $$w^3 = 1.$$

Now let $$a = x$$

and we have

$$x^3+b^3+c^3-3xbc = (x+b+c)(x+\omega^2 b+\omega c)(x+\omega b+\omega^2 c).$$

Hence the roots of $x^3 - 3bcx + b^3 + c^3 = 0$,

are $$-(b+c),\quad -(\omega^2 b+\omega c),\quad -(\omega b+\omega^2 c).$$

Again, let $bc = -\dfrac{q}{3}$, and $b^3 + c^3 = r$

and we get the roots of $x^3 + qx + r = 0$, to be

$$x = \sqrt[3]{A} + \sqrt[3]{B}$$
$$x = \alpha\sqrt[3]{A} + \alpha^2\sqrt[3]{B}$$
$$x = \alpha^2\sqrt[3]{A} + \alpha\sqrt[3]{B}$$

where

$$A = \left(-\frac{r}{2} + \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}\right),\quad B = \left(-\frac{r}{2} - \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}\right),\text{ and } \alpha^3 = 1$$

For the applications of determinants to continued fractions the student should consult Muir's papers (Proceedings of the Royal Society, Edinburgh, 1873-4), and also Scott's Determinants, Chap. XIII., and for additional applications any of the more advanced treatises on Determinants.

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